↓Lecture 5 [14.11.23]

3 | We now apply this algorithm twice, in the lab frame A and the rest frame A':

i | Rest frame A':

⁂ *Proper length* $\equiv$ ⁂ *Rest length* $:=$ Length of rod in A':

$$l_0 := \mathsf{LENGTH}(\mathcal{E}, t_0'; A') = |\vec{l}_0' - \vec{r}_0'| = |l_0' - r_0'| \tag{2.2}$$

with simultaneous clock events $(t_0', \vec{l}_0')_{A'} \in L_0$ and $(t_0', \vec{r}_0')_{A'} \in R_0$.

The time t_0' that we choose is irrelevant since the rod is (by definition) at rest in A'. Since the rod lies on the x'-axis, it is $\vec{l}_0' = (l_0', 0, 0)$ and $\vec{r}_0' = (r_0', 0, 0)$.

The subscript "0" in L_0 indicates that this is a specific event (coincidence class) we selected in A' to compute the length of the rod. It does *not* mean "as seen from the rest frame A'" or anything like that. Remember that coincidence classes in $\mathcal{E}$ are objective information!

ii | Lab frame A:

Length of moving rod in A:

$$l := \mathsf{LENGTH}(\mathcal{E}, t; A) = |\vec{l} - \vec{r}| \tag{2.3}$$

with simultaneous clock events $(t_l, \vec{l})_A \in L$ and $(t_r, \vec{r})_A \in R$ with $t_l = t_r = t$.

The time t that we choose might be irrelevant as well, but we do not know this yet.

¡! There is no reason to assume that the events L_0/R_0 chosen in A' to measure the length of the rod are identical to the events L/R used in A: $L_0 \neq L$ and $R_0 \neq R$ in general.

4 | How does l_0 relate to l?

i | In Section 1.5 we did a lot of hard work to compute the transformation φ which transforms the coordinates of an event in one inertial system into the coordinates of *the same event* in another inertial system. We identified the transformation as the Lorentz transformation:

$$\Lambda(A \xrightarrow{v_x} A') : [E]_A = (t, \vec{x}) = x \mapsto \Lambda_{v_x} x = x' = (t', \vec{x}') = [E]_{A'} \tag{2.4}$$

ii | So let us use this tool [namely Eq. (1.77)] to obtain the coordinates of the events L and R (used for the length measurement in A) in the rest frame A' of the rod:

$$[L]_{A'} = \begin{cases} ct_l' = \gamma\left(ct_l - \frac{v_x}{c} l_x\right) \\ l_x' = \gamma(l_x - v_x t_l) \\ l_y' = l_y \\ l_z' = l_z \end{cases} \quad \text{and} \quad [R]_{A'} = \begin{cases} ct_r' = \gamma\left(ct_r - \frac{v_x}{c} r_x\right) \\ r_x' = \gamma(r_x - v_x t_r) \\ r_y' = r_y \\ r_z' = r_z \end{cases} \tag{2.5}$$

Here we use $\vec{l} = (l_x, l_y, l_z)$ and $\vec{r} = (r_x, r_y, r_z)$. Since we declared that the rod is fixed on the x'-axis of A', and $\{e_L\} \in L$ and $\{e_R\} \in R$, it must be $l_y' = l_z' = r_y' = r_z' = 0$, and therefore $\vec{l} = (l_x, 0, 0)$ and $\vec{r} = (r_x, 0, 0)$. That is, the rod is not rotated by the boost and always lies on the x-axis of A as well. In particular: $l = |\vec{l} - \vec{r}| = |l_x - r_x|$.

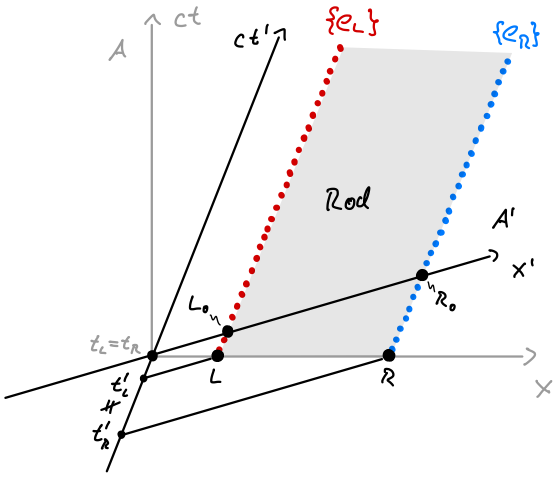
→ Two immediate conclusions:

a | In A' the two events L and R are *no longer simultaneous*:

$$t_l = t_r \text{ in } A \quad \text{but} \quad t'_l \neq t'_r \text{ in } A' \text{ (since } l_x \neq r_x\text{)}. \tag{2.6}$$

$$\boxed{\rightarrow \text{The simultaneity of events is } \textit{observer-dependent}.}$$

This ambiguity of simultaneity can be graphically illustrated in a spacetime diagram (for details on how to draw the (t', x')-axes in A: ➲ Problemset 2):



- As a side note, this calculation implies that not only is it generally *not* true that $L_0 = L$ and $R_0 = R$, it is actually *impossible* (at least for both pairs).
- In the sketch above, the "interior of rod"-events are painted gray. One is tempted to ask: Which "line" of these events *is* the rod? The counterintuitive answer is that this depends on the observer: For A-observers, *horizontal* lines of gray events make up "the rod", whereas for the A'-observer *tilted* lines are "the rod". It is actually more reasonable to think of the complete area of gray events as "the rod", just as the event type $\{e_L\}$ is "the left edge" of the rod. This suggests that our intuitive concept of the *instantaneous existence of extended objects* – which feels so natural to us – is, to some extend, misleading.

b | In A' the coordinate distance is different:

$$|l'_x - r'_x| \overset{t_l = t_r}{=} \gamma |l_x - r_x| \overset{v_x \neq 0}{\neq} |l_x - r_x| = l \tag{2.7}$$

¡! The time-dependence cancels so that the expressions are time-independent.

At this point, it is a bit premature to identify the left-hand side as the *rest length* l_0 of the rod because these are spatial coordinates of events that are *not simultaneous*! (Remember that the length of any object in any frame is defined as the coordinate distance of *simultaneous* events.)

However, since A' is (by definition) the *rest frame* of the rod, the position labels of the A'-clocks adjacent to the ends of the rod are the same for all events:

$$\left.\begin{aligned} l'_x &\overset{\{e_L\}\ni L}{=} l'_0 \\ r'_x &\overset{\{e_R\}\ni R}{=} r'_0 \end{aligned}\right\} \Rightarrow |l'_x - r'_x| = |l'_0 - r'_0| = l_0 \tag{2.8}$$

→ ⁂ *Length contraction* ≡ ⁂ *Lorentz contraction:*

A rod of rest length l_0 is *shorter* if measured from an inertial system in relative motion:

$$\boxed{l = l_0\sqrt{1-\frac{v^2}{c^2}} \overset{v\neq 0}{<} l_0} \tag{2.9}$$

- ¡! Due to isotropy, this result is true for any length of extended objects *in the direction of the boost*. A rod along the y'-axis, for example, is contracted according to Eq. (2.9) for a boost in y-direction, but not for a boost in x-direction.
- The rod is just a proxy for *any* physical object; the Lorentz contraction therefore affects all physical objects in the same way. The contraction is not a dynamical feature of the object itself (like a force that compresses the atomic lattice) but an intrinsic property of space(time).
- Note that we say above "if measured from …" and not "as viewed from …." This distinction is important: If you ask how you would *visually perceive* extended objects flying by (or how they look on a picture taken by a camera) you have to factor in that the photons bouncing of the object at different points take different times to reach your eye (our the camera sensor). If you do the math (➔ Problemset 3), this additional optical effect leads to the surprising result that 3D objects actually do *not look* "squeezed" but *rotated*. This implies in particular that a moving sphere still *looks* like a sphere and not like an ellipse (↑ *Penrose-Terrell effect* [40, 41], see also Ref. [42]).

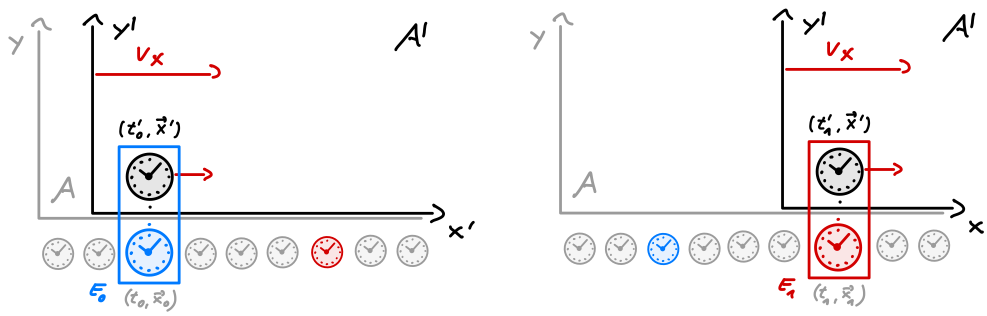
  You can experience this effect (among others) in the educational game "A Slower Speed of Light," which has been developed by the MIT Game Lab for educational purposes, and can be downloaded here for Windows, Mac, and Linux (➔ Problemset 3):

  ➔ Download "A Slower Speed of Light"

  You should always keep in mind, however, that this "looking" is *not* what we refer to as *observing* in RELATIVITY; the latter has been defined operationally as a measurement procedure at the beginning of this course.

2.2. Time dilation

1 | ⊲ Inertial systems $A \xrightarrow{v_x} A'$ and a clock $\vec{x}'$ at rest in A':



2 | ⊲ Two events:

$$A'\text{-Clock } \vec{x}' \text{ meets } A\text{-clock } \vec{x}_0: \quad (t'_0, \vec{x}')_{A'} \sim (t_0, \vec{x}_0)_A \in E_0 \tag{2.10a}$$

$$A'\text{-Clock } \vec{x}' \text{ meets } A\text{-clock } \vec{x}_1: \quad (t'_1, \vec{x}')_{A'} \sim (t_1, \vec{x}_1)_A \in E_1 \tag{2.10b}$$

¡! The two events E_0 and E_1 relate *three* different clocks: The single A'-clock $\vec{x}'$ and two *different* A-clocks $\vec{x}_0$ and $\vec{x}_1$.

3 | As for length, the concept of "duration" cannot be defined locally in spacetime. We therefore need an operational definition (algorithm) of "duration":

```
DURATION:
→ Input: Two events E_0 and E_1, Inertial system label K
← Output: Time interval Δt_K between events as measured in K
  1. Find (unique) clock event (t_0, x⃗_0)_K ∈ E_0.
  2. Find (unique) clock event (t_1, x⃗_1)_K ∈ E_1.
  3. Return Δt_K := t_1 − t_0.
```

Hopefully you agree that this is a reasonable definition of the duration (or time interval) between two events.

4 | We can now apply this algorithm to determine the time elapsed between E_0 and E_1:

$$\text{In } A': \quad \Delta t' = \mathtt{DURATION}(E_0, E_1; A') = t_1' - t_0' \quad \text{Measured by a single clock!} \tag{2.11a}$$

$$\text{In } A: \quad \Delta t = \mathtt{DURATION}(E_0, E_1; A) = t_1 - t_0 \quad \text{Measured by two clocks!} \tag{2.11b}$$

5 | <u>How does Δt relate to $\Delta t'$?</u>

i | Since $(t_0', \vec{x}')_{A'} \sim (t_0, \vec{x}_0)_A$ and $(t_1', \vec{x}')_{A'} \sim (t_1, \vec{x}_1)_A$, we can use the Lorentz transformation to translate between the coordinates:

$\xrightarrow{\text{Inverse of Eq. (1.77)}}$

Remember that $\Lambda_{\vec{v}}^{-1} = \Lambda_{-\vec{v}}$ because of reciprocity; the inverse Lorentz transformation can then be obtained by substituting $v_x \mapsto -v_x$:

$$[E_0]_A = \begin{cases} ct_0 = \gamma\left(ct_0' + \frac{v_x}{c}x'\right) \\ x_0 = \gamma(x' + v_x t_0') \end{cases} \quad \text{and} \quad [E_1]_A = \begin{cases} ct_1 = \gamma\left(ct_1' + \frac{v_x}{c}x'\right) \\ x_1 = \gamma(x' + v_x t_1') \end{cases} \tag{2.12}$$

We omit the other two coordinates since they are invariant anyway; the transformation of the spatial coordinate is also not necessary for the following derivation.

ii | Subtracting the equations for the time coordinate of both events yields:

$$c(t_1 - t_0) = \gamma c(t_1' - t_0') \tag{2.13}$$

Note that in the inverse Lorentz transformation Eq. (2.12) the position coordinate in A' is x' for *both* events because the *same* A'-clock takes part in both coincidences.

iii | ⁂ <u>*Time dilation:*</u>

→ The moving clocks in A' run slower than the stationary clocks in A:

$$\boxed{\Delta t = \frac{\Delta t_0}{\sqrt{1 - \frac{v^2}{c^2}}} \overset{v \neq 0}{>} \Delta t_0} \tag{2.14}$$

We renamed $\Delta t' \equiv \Delta t_0$ to emphasize the analogy to the *proper length* l_0:

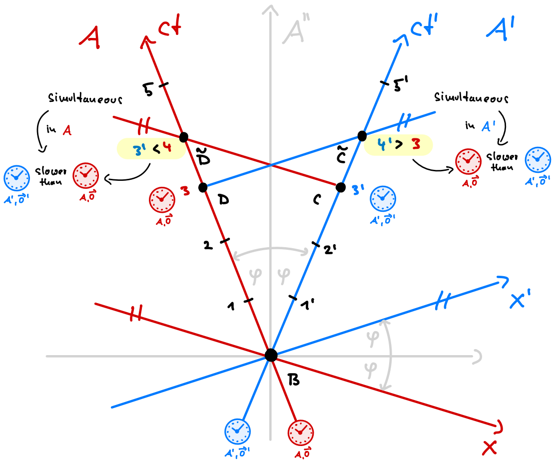
Δt_0: ⁂ *Proper time* elapsed in A' between E_0 and E_1
Δt: Time elapsed in A between E_0 and E_1

- The characteristic feature of the *proper time* Δt_0 between two (time-like separated) events E_0 and E_1 is that it can be measured by a *single* inertial clock that takes part in both events. All other time intervals must be measured by subtracting the reading of *two different* clocks. Eq. (2.14) tells you that these time intervals are always longer than the proper time Δt_0.
- ¡! Due to isotropy, our result above is true for boosts in any direction.

  Note that in the derivation above, we did *not* impose any special constraints on the positions of the clocks (except that they coincide pairwise at E_0 and E_1). In particular, we did not assume (despite the sketch suggesting this) that the clocks are located on the x/x'-axis. *All* clocks in A' are slowed down in the same way, irrespective of their location!
- This result does *not* contradict our assumption that all clocks are type-identical (= run with the same rate if put next to each other at rest) because the two events needed to compare the tick rate of moving clocks necessarily describe coincidences between *different* pairs of clocks.

6 | Relativity principle:

Because of the relativity principle SR time dilation must be completely *symmetrical*: The A'-clocks run slower compared to the A-clocks, and the A-clocks run slower compared to the A' clocks. That this is indeed that case (without being a clock "paradox") is best illustrated in a symmetric spacetime diagram:



The existence of the "median frame" A'' between $A \xrightarrow{v_x} A'$ can be easily shown with the addition for collinear velocities Eq. (1.70). This symmetric form of a spacetime diagram is sometimes called ↑ *Loedel diagram* [43] and makes the symmetry between inertial frames manifest; in particular, the units on the axes of A and A' are identical (they are not identical to the units of A'', tough). In this symmetric form, the t'-axis is orthogonal to the x-axis and the t-axis to the x'-axis. Note that because of the relativistic addition of velocities, it is $A'' \xrightarrow{\tilde{v}_x} A'$ and $A'' \xrightarrow{-\tilde{v}_x} A$ with $\tilde{v}_x = v_x \frac{\gamma}{1+\gamma}$ and $\tan(\varphi) = \frac{\tilde{v}_x}{c}$ (➔ Problemset 3). Only in the non-relativistic limit $v_x/c \to 0$ one finds $\tilde{v}_x = \frac{v_x}{2}$ as naïvely expected.

Note that due to the relativity of simultaneity, the two observers use *different* pairs of clock-events to decide which of the two origin clocks runs slower:

- For A the two clock events $\tilde{D}$ and C are simultaneous such that one has to conclude that the (blue) A'-clock runs slower than the (red) A-clock.

- By contrast, for the observer A' the two events D and $\tilde{C}$ are simultaneous such that one has to conclude that the (red) A-clock runs slower than the (blue) A'-clock.

It is evident from the diagram that there is no disagreement about coincidences of events (or readings of clocks). It is just the observer-dependent concept of simultaneity that leads to the seemingly "paradoxical" reciprocity of time dilation.

7 | Experiments:

- *Muon decay [44]:*

  Muons quickly decay into electrons (and neutrinos):

  $$\mu^- \to e^- + \nu_\mu + \bar{\nu}_e\,. \tag{2.15}$$

  This decay can be readily observed in storage rings of particle colliders like CERN. The lifetime of muons *at rest* (measured by clocks in an inertial laboratory frame) is $\tau_\mu^0 \approx 2.1948(10)\,\mu\mathrm{s}$. However, the lifetime of muons in flight (close to the speed of light) is measured to be $\tau_\mu \approx 64.368(29)\,\mu\mathrm{s}$, i.e., much longer! If one carefully takes into account the speed of the muons and additional experimental imperfections, this result fits Eq. (2.14) with deviations of only $\sim 0.1\,\%$ [44].

  Notes:

  - In the rest frame of the flying *muons* one would measure the usual lifetime $\tau_\mu^0 \approx 2.1948(10)\,\mu\mathrm{s}$. However, in this frame, the *laboratory* is *Lorentz contracted* such that the muon reaches exactly the same point in space where it decays in this "shorter" lifetime. Note how time-dilation and Lorentz contraction provide different explanations for the same experimental obervation.
  - One can also use different particle species to study time dilation, for example *pions* (a sort of meson, i.e., a hadron with one quark and one antiquark) [45].

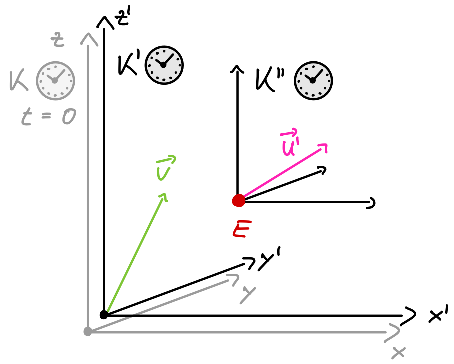
- *Hafele-Keating experiment [46, 47]:*

  In 1971, J.C. Hafele and R. E. Keating took four Cesium atomic clocks along commerical jet flights around the globe twice: once eastward and once westward. Compared to a reference clock on the ground, the clocks on the eastward flight lost on average $\sim 59\,\mathrm{ns}$ (= they ran slower) and the clocks on the westward flight gained $\sim 273\,\mathrm{ns}$ (= they ran faster). To understand this qualitatively, note that the reference clock on the ground is *rotating* (together with earth) and therefore is *not* an inertial clock. Therefore imagine an (approximately) inertial reference system flying along earth around the sun, and from this system look down on the north pole; earth is now slowly rotating beneath you. From this inertial system, the eastward flight has higher velocity than the reference clock, which, in turn, has higher velocity than the westward flight. Thus you find that the eastward clock runs slower than the reference clock which runs slower than the westward clock (this is also true if the clocks are accelerated,→ *below*). These theoretical considerations are explained in [46].

2.3. Addition of velocities

Details: ➔ Problemset 2

1 | $\sphericalangle$ Particle moving with $\vec{u}' = \frac{\mathrm{d}\vec{x}'}{\mathrm{d}t'}$ in system K' and inertial system K with $K \xrightarrow{\vec{v}} K'$:



2 | Velocity $\vec{u}$ in K:

$$\boxed{\vec{u} = \frac{\mathrm{d}\vec{x}}{\mathrm{d}t} \equiv \vec{v} \oplus \vec{u}' \stackrel{\circ}{=} \frac{1}{1+\frac{\vec{v}\cdot\vec{u}'}{c^2}}\left[\vec{v} + \frac{\vec{u}'}{\gamma_v} + \frac{\gamma_v}{c^2(1+\gamma_v)}(\vec{u}'\cdot\vec{v})\,\vec{v}\right]} \tag{2.16}$$

Proof: Use Eq. (1.75) (➲ Problemset 2).

¡! The relativistic addition of velocities $\oplus$ is in general not commutative ($\vec{v}\oplus\vec{u} \neq \vec{u}\oplus\vec{v}$) nor associative [$\vec{v}\oplus(\vec{u}\oplus\vec{w}) \neq (\vec{v}\oplus\vec{u})\oplus\vec{w}$]. As you can easily see from Eq. (2.16), it is also not linear: $(\lambda\vec{v})\oplus(\lambda\vec{u}) \neq \lambda(\vec{v}\oplus\vec{u})$. Be careful: There are different notations (in particular: orderings) used in the literature.

3 | ⊲ Non-relativistic limit ($c\to\infty \Rightarrow \gamma_v \to 1$):

$$\lim_{c\to\infty} \vec{v}\oplus\vec{u}' = \lim_{c\to\infty} \vec{u}'\oplus\vec{v} = \vec{v}+\vec{u}' \tag{2.17}$$

→ Galilean addition of velocities

4 | Special case: $\vec{v} = (v_x, 0, 0)$:

$$u_x \stackrel{\circ}{=} \frac{v_x + u'_x}{1+\frac{v_x u'_x}{c^2}}\,,\quad u_y \stackrel{\circ}{=} \frac{u'_y/\gamma_v}{1+\frac{v_x u'_x}{c^2}}\,,\quad u_z \stackrel{\circ}{=} \frac{u'_z/\gamma_v}{1+\frac{v_x u'_x}{c^2}}\,. \tag{2.18}$$

¡! Note that also the transverse components of $\vec{u}'$ are modified, but in a different way than the collinear component u'_x. For $\vec{u}' = (u'_x, 0, 0)$ we get our previous result for collinear velocities Eq. (1.70) back.

5 | Thomas-Wigner rotation [48, 49]:

Remember that for *collinear* addition of velocities the concatenation of two boosts yields another boost: $\Lambda_{v_x}\Lambda_{u_x} = \Lambda_{w_x}$ [recall Eq. (1.57)].

As a straightforward (but tedious) calculation using two general boosts Eq. (1.75) shows, this is *not* true in general: $\Lambda_{\vec{v}}\Lambda_{\vec{u}} \neq \Lambda_{\vec{w}}$ with $\vec{w} = \vec{u}\oplus\vec{v}$. Rather one finds

$$\Lambda_{\vec{v}}\Lambda_{\vec{u}} = \Lambda_{\vec{u}\oplus\vec{v}}\Lambda_{R(\vec{u},\vec{v})} \tag{2.19}$$

with the ⁂ *Thomas-Wigner rotation* $R(\vec{u},\vec{v}) \in \mathrm{SO}(3)$ (we omit the expliclit form of $R(\vec{u},\vec{v})$ here).

This is not in contradiction with our general addition for velocities above because there we were only interested in the velocity of a moving particle (which you can identify with the origin of its rest frame K''); we completely ignored the *axes* of K''. The Thomas-Wigner rotation tells you that the concatenation of two *pure* boosts is *not* a pure boost in general.

2.4. Proper time and the twin "paradox"

1 | $\triangleleft$ Time-like trajectory $\mathcal{P} \subseteq \mathcal{E}$ of a spaceship with departure $D \in \mathcal{P}$ and arrival $A \in \mathcal{P}$.

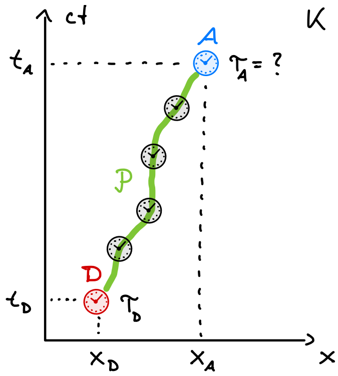
$\triangleleft$ Coordinate parametrization $\vec{x}(t)$ of $\mathcal{P}$ in system K with

$$\text{departure } [D]_K = (t_D, \vec{x}_D) \quad \text{and} \quad \text{arrival } [A]_K = (t_A, \vec{x}_A): \tag{2.20}$$

Formally, $\mathcal{P}$ is a set of coincidence classes parametrized in K by the clock events $(t, \vec{x}(t))_K$:

$$\mathcal{P} = \{ [(t, \vec{x}(t))_K] \,|\, t \in [t_D, t_A] \} \subseteq \mathcal{E}. \tag{2.21}$$

This suggests the formal notation $[\mathcal{P}]_K = (t, \vec{x}(t))$.



2 | Thought experiment:

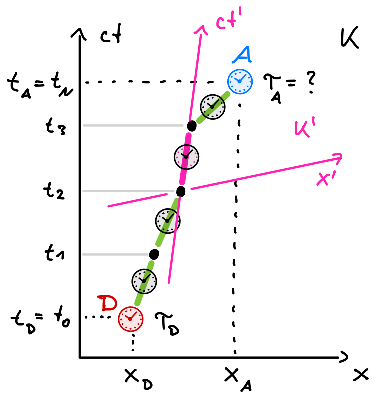
The spaceship takes a clock along and resets it to $\tau_D = \tau(t_D)$ at departure D.

What is the reading $\tau_A = \tau(t_A)$ of the clock at arrival A?

We assume that the clock in the spaceship is type-identical to the clocks used for inertial observers.

3 | *Idea:*

Approximate the trajectory by a *polygon* of N segments $i = 1, \dots, N$ separated by time steps t_i (with $t_0 := t_D$ and $t_N := t_A$):



i | Let $\Delta t_i := t_{i-1} - t_i$ and $\Delta \vec{x}_i := \vec{x}(t_{i-1}) - \vec{x}(t_i)$

For each segment, there is an inertial frame K' with a t'-axis that follows the spacetime segment (because all segments are time-like!). This is the instantaneous rest frame of the spaceship where the clock in the spaceship and the origin clock of K' are at the same place and at rest relative to each other. Since the clocks are type-identical, the time $\Delta\tau_i$ accumulated by the spaceship clock on this segment is identical to the time $\Delta t_i'$ elapsed for the origin clock of K' on this segment: $\Delta\tau_i = \Delta t_i'$. This time is equal to the spacetime interval $(\Delta s_i')^2 = (c\Delta t_i')^2 - 0$ because the origin clock is at rest in K' (so that $\Delta\vec{x}_i' = \vec{0}$). But remember that the spacetime interval $(\Delta s_i')^2$ is Lorentz invariant so that we can calculate *the same number* in any inertial system: $(\Delta s_i')^2 = (\Delta s_i)^2 = (c\Delta t_i)^2 - (\Delta\vec{x}_i)^2$.

In summary, on the ith interval, the spaceship clock accumulates the time

$$\Delta\tau_i = \frac{\Delta s_i}{c} := \frac{\sqrt{\Delta s_i^2}}{c} = \frac{\sqrt{(c\Delta t_i)^2 - (\Delta\vec{x}_i)^2}}{c} = \Delta t_i \sqrt{1 - \frac{(\Delta\vec{x}_i/\Delta t_i)^2}{c^2}} \tag{2.22}$$

The above chain of arguments provided us with a *physical interpretation* for the Lorentz invariant spacetime interval $(\Delta s)^2 > 0$ of time-like separated events: It measures (up to a factor of c) the time accumulated by an inertial (= unaccelerated) clock that takes part in both events.

ii | Continuum limit $N \to \infty$ ($v(t) := |\vec{v}(t)| = |\dot{\vec{x}}(t)|$):

$$\boxed{\mathrm{d}\tau = \frac{\mathrm{d}s}{c} = \mathrm{d}t\sqrt{1 - \frac{\dot{\vec{x}}(t)^2}{c^2}} \quad\Leftrightarrow\quad \frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma_{v(t)}} \tag{2.23}$$

Note that this is just an infinitesimal version of the time-dilation formula Eq. (2.14) with $\Delta t \to \mathrm{d}t$ and $\Delta t_0 \to \mathrm{d}\tau$.

Since $(\Delta s)^2 = (\Delta s')^2$ is *Lorentz invariant*:

$$K \xrightarrow{\Lambda} K': \quad \mathrm{d}t\sqrt{1 - \frac{\dot{\vec{x}}(t)^2}{c^2}} = \frac{\mathrm{d}s}{c} = \frac{\mathrm{d}s'}{c} = \mathrm{d}t'\sqrt{1 - \frac{\dot{\vec{x}}'(t')^2}{c^2}} \tag{2.24}$$

You can check this also explicitly using the Lorentz transformation Eq. (1.75).