↓Lecture 4 [07.11.23]

18 | Experiments (in particular: the validity of Maxwell equations) show:

$$\boxed{v_{\max} = c = 299\,792\,458\,\mathrm{m\,s^{-1}}} \tag{1.74}$$

Note that since 1983 the value of c in the international system of units (SI) is *exact* by definition.

A. Einstein incorporated this insight in §2 of Ref. [9] as his second postulate:

> *2. Jeder Lichtstrahl bewegt sich im "ruhenden" Koordinatensystem mit der bestimmten Geschwindigkeit V, unabhängig davon, ob dieser Lichtstrahl von einem ruhenden oder bewegten Körper emittiert ist.*

Note that at the time it was conventional to denote the speed of light with a capital V. The convention switched to our now standard lower-case c just a few years later. For more historical background:

➔ https://math.ucr.edu/home/baez/physics/Relativity/SpeedOfLight/c.html

We can condense this into:

§ Postulate 5: Constancy of the speed of light SL

The speed of light is independent of the inertial system in which it is measured.

Comments:

- *If* you take the validity of the Maxwell equations for granted, then $v_{\max} = c < \infty$ (and thereby SL) follows immediately from the relativity principle SR because then the Maxwell equations must be valid in all inertial systems. But you've learned in your course on electrodynamics that the wavelike solutions of these equations always propagate with group velocity c in vacuum. This is only possible if the speed of light plays the role of the limiting velocity: $v_{\max} = c$.

  Einstein acknowledges as much at the beginning of Ref. [10]. However, SL is empirically weaker than claiming the validity of Maxwell's equations (after all, there could be alternative equations that also predict the velocity c of wavelike solutions). At the time when Einstein formulated SL in [9], he also worked on the photoelectric effect (another of his *annus mirabilis* papers [37]). The postulation of "quanta of light" is the foundation of quantum mechanics, but cannot be explained by Maxwell's equations. It is therefore reasonable to assume that Einstein didn't want to rely on the validity of this specific theory when formulating his SPECIAL RELATIVITY. He therefore opted for the empirically weaker (but still sufficient) assumption SL.

- If you derive the transformation φ using *both* postulates SR and SL the derivation is shorter (see e.g. [1] or [4]); one then of course doesn't find the Galilei transformations as an option. Note, however, that the relativity principle SR is a reasonable and intuitive starting point that doesn't need much convincing (after all, we witness the relativity of Newtonian mechanics in our everyday life). By contrast, the speed of light postulate SL clashes directly with our everyday experience (how velocities add up, that is). Through our elaborate derivation we learned how much is already implied by the simple, reasonable assumption of relativity. We only had to check whether there is any evidence of a finite maximum velocity $v_{\max}$. The counterintuitive feature that this velocity is the same viewed from all inertial systems was then a necessary conclusion from our derivation.

† Note: Finite speed of causality (Locality)

Another insight from our SR-based derivation of the Lorentz transformation is that the formulation of the speed-of-light postulate SL is conceptually misleading:

- The constant $v_{\max}$ and its role as a maximum velocity followed *without* referring to light (or electrodynamics) in any way!

  Put bluntly: SPECIAL RELATIVITY is *not* about the "strange behavior" of light!

- The relevant speed for SPECIAL RELATIVITY is the *speed of causality*: How fast can information travel, i.e., one event affect another. $v_{\max}$ is the maximum speed of *causal interactions*, irrespective of the mediator of these interactions.

  In our world, the fastest and most salient information carrier just happens to be the electromagnetic field ("light"). For example, to synchronize our clocks with light signals, it wasn't the light *per se* we were interested in; we just used it as carrier of information to correlate the clocks.

- Given the relativity principle SR and our derivation in Section 1.4, we showed that there are only two possibilities: (1) There is *no* upper bound on velocities (Galilean symmetry) or (2) there *is* such an upper bound $v_{\max}$ (Lorentz symmetry). In the latter case, every signal that propagates with $v_{\max}$ in some frame automatically does so in all inertial systems. (Which immediately leads to the counterintuitive conclusion, akin to SL, that there are signals the velocity of which does not depend on the velocity of the observer.)

- We could replace SL therefore by the (empirically weaker) postulate that there are *no instantaneous actions* at a distance (this is essentially a statement about *locality*). This modified postulate implies the existence of a maximal velocity $v_{\max} < \infty$ which, in turn, selects the Lorentz transformation as the correct symmetry. That $v_{\max} = c$ is then a fact to be discovered by experiments.

- It turns out that *everything with vanishing rest mass* travels at this maximum speed $v_{\max} = c$. Since photons are the only elementary particles that are massless and can be easily detected, we just happen to refer to this maximum velocity as "speed of light."

  For example: Without Higgs symmetry breaking, the $W^{\pm}$ and Z bosons of the weak interaction are massless and would propagated with light velocity, just as the photon (the weak interactions would then be no longer "weak"). For a long time it was believed that neutrinos are massless as well, and therefore would also propagate with the speed of light (today we know that they have a very tiny mass).

19 | Special Lorentz transformations = Lorentz boosts:

Now that everything is settled, let us write down our final result in their conventional form.

¡! These are not the most general (homogeneous) Lorentz transformations since we omit rotations, parity and time inversion. We will discuss the structure of the full homogeneous Lorentz group and its inhomogeneous generalization (→ *Poincaré group*) later. To discuss the "fancy" phenomena of SPECIAL RELATIVITY, the transformations below are sufficient.

i | Boost in arbitrary directions ($\vec{v} = v\hat{v}$ with $\hat{v} \equiv \vec{v}/|\vec{v}|$):

$$\boxed{\Lambda(K \xrightarrow{\vec{v}} K') : \quad \begin{cases} ct' = \gamma\left(ct - \beta\vec{x}\cdot\hat{v}\right) \\ \vec{x}' = \vec{x} + (\gamma - 1)(\vec{x}\cdot\hat{v})\cdot\hat{v} - \gamma\vec{v}t \end{cases}} \tag{1.75}$$

(Since we now settled on Lorentz transformation for φ, we write $\varphi = \Lambda$ henceforth.)

with $\beta \equiv v/c$ and the *Lorentz factor*

$$\boxed{\gamma_v \equiv \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1-\beta^2}}\,.} \tag{1.76}$$

ii | Special case: Boost in x-direction ($\vec{v} = v_x\hat{x}$):

$$\boxed{\Lambda(K \xrightarrow{v_x} K') : \quad \begin{cases} ct' = \gamma\left(ct - \frac{v_x}{c}x\right) \\ x' = \gamma(x - v_x t) \\ y' = y \\ z' = z \end{cases}} \tag{1.77}$$

20 | <u>State of affairs:</u>

Now that we know the spacetime symmetry φ of reality, we have quite a to-do list:

- We will have to *modify* Newton's equations to replace their Galilean by a Lorentz symmetry, without changing their predictions for small velocities $v \ll c$ (↓ *correspondence principle*).

  → Relativistic mechanics

- We can keep the Maxwell equations in their current form ☺.

  Note that we still have to check that they are really Lorentz covariant (➔ Problemset ?)!

  In the end we will come up with a neat notation that allows us to rewrite (not modify!) the Maxwell equations in a compact form to make their Lorentz symmetry apparent.

- Similar to classical mechanics, we will have to replace the *Schrödinger equation* in quantum mechanics by a modified version with Lorentz symmetry.

  → Relativistic quantum mechanics (Klein-Gordon and Dirac equation)

But before we do all the heavy work:

Simple implications of this transformation? (→ *below* and next lectures)

With "simple" we refer to implications that follow without imposing a model-specific dynamics (= equation of motion). We will refer to these implications as *kinematic* because they follow from fundamental constraints on the degrees of freedom of all relativistic theories.

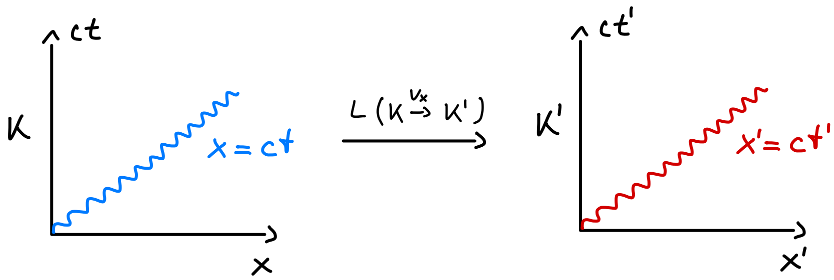
1.6. Invariant intervals and the causal partial order of events

1 | $\sphericalangle$ Trajectory of a light signal in x-direction in K:

$$x(t) = ct, \;\; y = 0, \;\; z = 0 \tag{1.78}$$

Trajectory of the same signal in K' with $K \xrightarrow{v_x} K'$:

$$x'(t') = ct', \;\; y' = 0, \;\; z' = 0 \tag{1.79}$$



This follows from our previous discussion: signals propagating with $c = v_{\max}$ do so in all inertial systems!

You can also simply calculate this using the Lorentz boost Eq. (1.77):

$$ct' = \gamma\left(ct - \tfrac{v_x}{c}ct\right) \tag{1.80a}$$

$$\text{and} \quad x' = \gamma(ct - v_x t) = ct' \,. \tag{1.80b}$$

→

$$(ct)^2 - x^2 = 0 = (ct')^2 - (x')^2 \quad \text{is a } \textit{frame-independent} \text{ quantity.} \tag{1.81}$$

Note that the separate summands [$(ct)^2$ etc.] are *not* frame-independent!

This finding motivates the definition of the …

2 | ⁂ *Spacetime interval*:

Details: ➲ Problemset 2

$\sphericalangle$ Two events $E_1 \ni (t_1, \vec{x}_1)_K$ and $E_2 \ni (t_2, \vec{x}_2)_K$ with temporal and spatial separation

$$(\Delta t)_K := t_1 - t_2 \quad \text{and} \quad (\Delta\vec{x})_K := \vec{x}_1 - \vec{x}_2 \,. \tag{1.82}$$

Then the *spacetime interval* between E_1 and E_2 is denoted $(\Delta s)^2 \equiv \Delta s^2$ and defined as

$$\boxed{(\Delta s)^2 := (c\Delta t)_K^2 - (\Delta\vec{x})_K^2 \,.} \tag{1.83}$$

We omit the subscript K from Δs because it is frame-independent (→ *next*).

In our example above it was $\Delta t = t - 0$ and $\Delta\vec{x} = (x - 0, 0 - 0, 0 - 0)$, i.e., we considered the interval between the event in the origin $x_O = (0, \vec{0})$ and the events along the trajectory $(ct, x(t), 0, 0)$ of the light signal.

3 | The importance of Δs^2 stems from the following fact:

> The spacetime interval Δs^2 is independent of the frame in which it is calculated.

This means that given two events, all observers agree on the numerical value of the interval Δs^2 between these two events.

Proof: Use Eq. (1.75) to calculate (Details: ➔ Problemset 2)

$$(ct')^2 = \left[\gamma\left(ct - \beta\vec{x}\cdot\hat{v}\right)\right]^2 \tag{1.84a}$$

$$(\vec{x}')^2 = \left[\vec{x} + (\gamma-1)(\vec{x}\cdot\hat{v})\cdot\hat{v} - \gamma\vec{v}t\right]^2 \tag{1.84b}$$

$$\Rightarrow \quad (ct')^2 - (\vec{x}')^2 \stackrel{\circ}{=} (ct)^2 - (\vec{x})^2 + \underbrace{\dots}_{=0} \tag{1.84c}$$

Note that we do not have to do the computation for two events and an interval Δt and $\Delta\vec{x}$ since the special Lorentz transformations are *linear*.

This proves the invariance under *special* Lorentz transformations (= Lorentz boosts). It is easy to see that the invariance is also valid for inhomogneous shifts in time and space (these drop out in the intervals Δt etc.) and spatial rotations Λ_R [since $(\Delta\vec{x})^2$ is clearly invariant under rotations]. We will come back to this when we discuss the structure of the Lorentz group in more detail (→ *later*).

4 | Two events E_1 and E_2 are in one of three possible (frame-independent) relations:

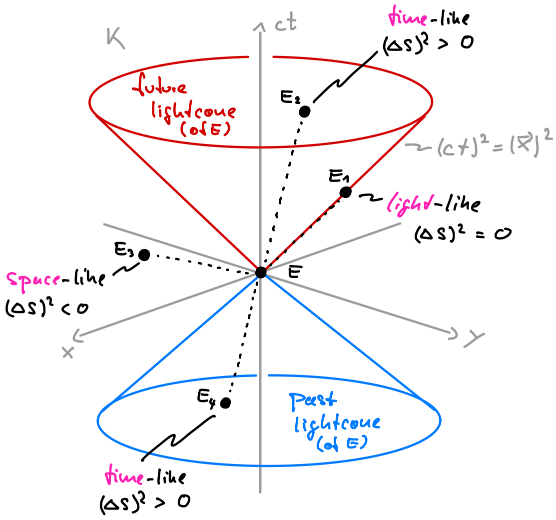
$$\Delta s^2 \begin{cases} > 0 & E_1 \text{ and } E_2 \text{ are ⁂ \textit{time-like} separated} \\ = 0 & E_1 \text{ and } E_2 \text{ are ⁂ \textit{light-like} separated} \\ < 0 & E_1 \text{ and } E_2 \text{ are ⁂ \textit{space-like} separated} \end{cases} \tag{1.85}$$

Note that Δs^2 can be *negative* so that Δs^2 should be read as a symbol rather than defining an imaginary number Δs. For the special case of *time-like* intervals, however, Δs^2 indeed defines a real number $\Delta s = \sqrt{\Delta s^2}$ which we will later relate to the time measured by moving clocks (the so called *proper time*).

All events that are light-like separated from an event E (*wlog* in the origin) satisfy

$$\Delta s^2 = 0 \quad \Leftrightarrow \quad (ct)^2 = (\vec{x})^2 \quad \Leftrightarrow \quad |ct| = |\vec{x}| \tag{1.86}$$

which determines the ⁂ *light cone* of E:



Here we show the light cone of an event E in a space time with *two* spatial dimensions x and y. The light cone in our $3+1$ dimensional space time is a higher-dimensional generalization which obeys the same equations.

- Time-like events satisfy $\Delta s^2 > 0 \quad \Leftrightarrow \quad |ct| > |\vec{x}|$ which characterizes the (disconnected) interior of the light cone. The manifold with $ct > |\vec{x}| \geq 0$ is called ⁂ *future light cone* (of E) whereas the events with $-ct > |\vec{x}| \geq 0$ make up the ⁂ *past light cone* (of E).
- Space-like events satisfy $\Delta s^2 < 0 \quad \Leftrightarrow \quad |ct| < |\vec{x}|$ which characterizes the (connected) spacetime volume outside the light cone.

5 | Causality:

The importance of the threefold classification of spacetime intervals stems from the following observations.

i | ⊲ Actions of (homogeneous) Lorentz transformations:

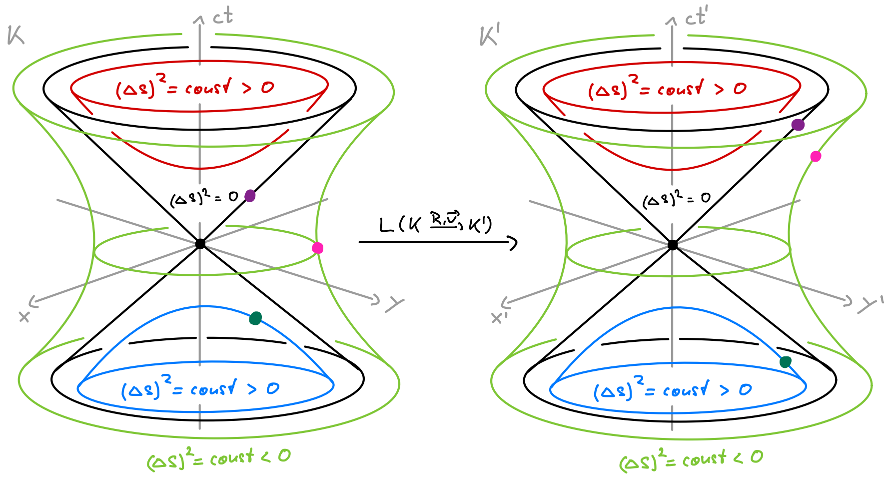
Since Δs^2 is invariant under Lorentz transformations, the manifold of events characterized by a specific value $\Delta s^2 = \pm C$ ($C \geq 0$) must be mapped onto itself under these transformations: Events on these hyperbolic manifolds cannot leave their manifolds under Lorentz transformations.

Invariant hyperbolae:

$$\text{time-like:} \quad \Delta s^2 = C > 0 \quad \Rightarrow \quad ct = \pm\sqrt{C + |\vec{x}|^2} \tag{1.87a}$$

$$\text{light-like:} \quad \Delta s^2 = C = 0 \quad \Rightarrow \quad ct = \pm|\vec{x}| \tag{1.87b}$$

$$\text{space-like:} \quad \Delta s^2 = -C < 0 \quad \Rightarrow \quad ct = \pm\sqrt{|\vec{x}|^2 - C} \tag{1.87c}$$



This picture leads immediately to two conclusions:

ii | $\triangleleft$ Two *distinct* events $E_1 \ni (t_1, \vec{x}_1)_K$ and $E_2 \ni (t_2, \vec{x}_2)_K$ with coordinates in K:

- If $\Delta s^2 \geq 0$ (= time-like or light-like), then

$$\text{either} \quad \forall_K : \ (t_1)_K > (t_2)_K \quad \text{or} \quad \forall_K : \ (t_1)_K < (t_2)_K \,. \tag{1.88}$$

This means that for time-like or light-like separated events all observers agree on their temporal ordering! Note that they do not necessarily agree on the time $(t_1)_K - (t_2)_K$ elapsed between the two events.

Proof: Assume $(t_1)_A < (t_2)_A$ and $(t_1)_B > (t_2)_B$ for two inertial systems A and B. Because of the continuity of Lorentz transformations there must exist a frame C with $(t_1)_C = (t_2)_C$. But in this frame $(\Delta s)^2_C = -(\Delta \vec{x})^2_C \geq 0$ such that $(\vec{x}_1)_C = (\vec{x}_2)_C$ and therefore $E_1 = E_2$ (which contradicts our assumption that the two events are distinct).

Proof by picture!

- If $\Delta s^2 < 0$ (= space-like), then

$$\exists_{A,B} : \ (t_1)_A > (t_2)_A \quad \text{and} \quad (t_1)_B < (t_2)_B \,. \tag{1.89}$$

This means that for space-like separated events there are always observers who see E_1 happening before E_2 while other observers see E_1 happening after E_2. The temporal order of space-like separated events is therefore observer-dependent!

Proof: ➔ Problemset ?

Proof by picture!

iii | Conventional relation of time order and causality:

$$E_1 \text{ can causally affect } E_2 \ \Rightarrow \ E_1 \text{ happens before } E_2 \tag{1.90}$$

Since causality should be an objective, observer-independent fact, and we just showed that only time- and light-like separated events have an observer-independent temporal order, it is reasonable to define the following …

… ↓ *(strict) partial order* $\prec$ on the set $\mathcal{E}$ of events:

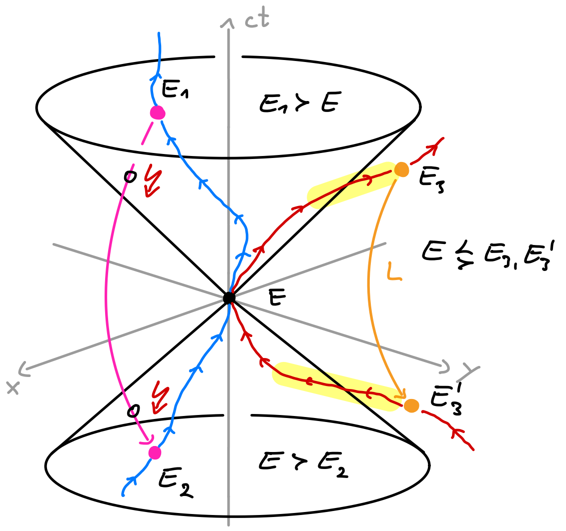
$$E_1 \prec E_2 \quad :\Leftrightarrow \quad \Delta s^2 \geq 0 \quad \text{and} \quad t_1 < t_2 \ : \quad \text{“}E_1 \text{ can affect } E_2\text{”} \tag{1.91}$$

$$E_1 \succ E_2 \quad :\Leftrightarrow \quad \Delta s^2 \geq 0 \quad \text{and} \quad t_1 > t_2 \ : \quad \text{“}E_2 \text{ can affect } E_1\text{”} \tag{1.92}$$

This is a *partial* order because for $\Delta s^2 < 0$ there is *no* relation between E_1 and E_2 (we denote this by $E_1 \between E_2$).

To be a partial order, one has to show *irreflexivity* (which is trivial since $t < t$ is not true) and *transitivity*. To show transitivity, show that $\Delta s_{1,2}^2 \geq 0$ and $\Delta s_{2,3}^2 \geq 0$ together with $t_2 > t_1$ and $t_3 > t_2$ implies $\Delta s_{1,3}^2 \geq 0$ and $t_3 > t_1$ (use the triangle inequality).

iv | This definition of causality is consistent with our previous findings that no signal can travel faster than the speed of light c:



- $E \prec E_1$: There exists a signal trajectory $\vec{x}(t)$ with $\left|\frac{\mathrm{d}\vec{x}(t)}{\mathrm{d}t}\right| \leq c$ connecting the two events (blue in the sketch).
- $E \between E_3$: Any trajectory $\vec{x}(t)$ connecting the two events (red in the sketch) has some segment with $\left|\frac{\mathrm{d}\vec{x}(t)}{\mathrm{d}t}\right| > c$ (yellow in the sketch). Since this is physically impossible, there is no signal of any kind that can mediate causal influence from E to E_3 (and vice versa).

  This follows from an application of (a generalization of) the ↓ *mean value theorem*.

6 | Since the causal structure $(\mathcal{E}, \prec)$ is observer independent:

> There is *no* relativity of causality in SPECIAL RELATIVITY!

If one observer states that E_1 can causally affect E_2, then *all* observers will agree on this statement.

7 | *Fun fact:*

If one starts from the causal structure $(\mathcal{E}, \prec)$ and derives the group of ↑ *causality-preserving automorphisms* Φ,

$$E_1 \prec E_2 \quad \Leftrightarrow \quad \Phi(E_1) \prec \Phi(E_2)\,, \tag{1.93}$$

one again finds the homogeneous Lorentz transformations (boosts & rotations) that we constructed above (plus space-inversion, spacetime dilations and translations), see Ref. [38] for more details. Most interestingly, for the proof neither a continuity assumption on Φ nor a topology on $\mathcal{E}$ is required; all this follows (at least in $2+1$ spacetime dimensions and more) from the partial order $\prec$.

1.7. ‡ Relativity, compressibility, and the anthropic principle

The statements in this section are not specific to Einstein's relativity principle SR.

1 | Relativity principles …

- … are statements about (the existence of) *symmetries* of spacetime.
- … imply the versatility of models to predict events *from many viewpoints*.
- … are statements about an *a priori* unnecessary *simplicity* of nature.

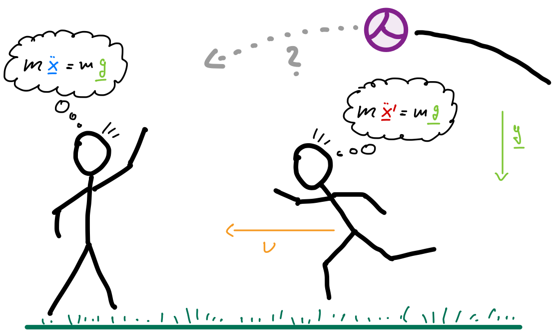
2 | Imagine a world *without any* relativity principle:

The equations (models) that capture physical laws faithfully are different from frame to frame.

→ Your brain must learn arbitrary many *different* models adapted to all possible reference frames to anticipate the future in all situations.

→ Biologically impossible (your brain capacity is finite, building models is expensive)

3 | Example: Catching balls:



Notice that most reference frames that we naturally encounter are (approximately) inertial only in x and y direction (the axes that are locally parallel to earth's surface) and constantly accelerated in z direction (the axis perpendicular to earth's surface; the acceleration is $g \approx 9.81\,\mathrm{m/s^2}$). The non-relativistic symmetries that relate these frames are a *subgroup* of the full Gallilei group (excluding rotations around the x and y axes as well as "large" translations). Our brain contains only models for these frames (equipped with Cartesian coordinates). Have you ever tried throwing or catching a ball in frames with acceleration in x or y directions (like a centrifuge)?

➔ YouTube Video: The artificial gravity lab (Tom Scott)

Note that it is not *impossible* to train specific models for other frames to which the relativity principle of our everyday experience does not apply (after some practice you *can* throw and catch balls in a centrifuge of constant angular velocity). But this is just *one* additional model and even this is not implemented in our brains by default!

4 | Relativity principle

→ Descriptions of natural phenomena are highly *compressible*.

→ Only *few* models (equations) are necessary to anticipate the future.

5 | Anthropic principle:

Question: Why are there spacetime symmetries / relativity principles in the first place?

Answer: Because if there were none, evolution would most likely be impossible, hence we would be unable to ask the question.

Note that evolution relies on the somewhat reliable proliferation of information over time. This seems only possible if the individuals carrying this information survive. Surviving in environments with life-threatening phenomena (thunderstorms, predators, ...) relies on its (approximate) *predictability* by (approximate) *models* that are learned evolutionary and/or by experience.

For this argument to work some form of "ensemble interpretation" of reality is required (e.g. ↑ *multiverses*) [39].

2. Kinematic Consequences

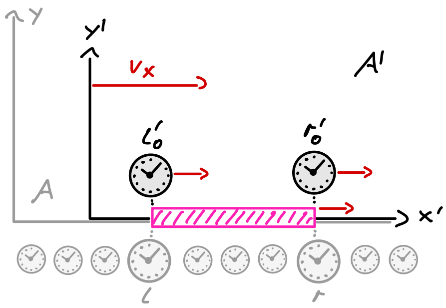
In this chapter we study implications of the special Lorentz transformations Eq. (1.75) and Eq. (1.77) that follow without imposing a model-specific dynamics (= equations of motion). We refer to these implications as *kinematic* because they follow from fundamental constraints on the degrees of freedom of all relativistic theories. The phenomena we will encounter are therefore *features of spacetime itself* – and not of some entities that live on/in (or couple to) spacetime.

¡! The phenomena we will encounter are *not* "illusions" (in the sense that we "see" things differently than they "really are"). Remember that we precisely defined what we mean by observers/reference frames; in particular, we emphasized that we do not "look" at anything, we *measure* events in a systematic way, using a well-defined structure called ← *inertial system*. All phenomena we will encounter are derived from and to be understood in this operational, physically meaningful context.

2.1. Length contraction and the Relativity of Simultaneity

1 | $\triangleleft$ Inertial systems $A \xrightarrow{v_x} A'$ with rod on x'-axis and at rest in A':

Remember that $A \xrightarrow{v_x} A'$ denotes a boost in x-direction with v_x (as measured in A) where the spatial axes of both A and A' coincide at $t = 0$:



In such situations, we refer to A' as the ⁂ *rest frame* of the rod and A as the ⁂ *lab frame* (some call A the ⁂ *stationary frame*). In the following, coordinates of events in the inertial system A' are marked by primes.

2 | First, we have to define what we mean by the "lenght" of an object:

"Length" is an intrinsically non-local concept. It is not something you can measure or define at a single point in space. Consequently, there are no "length-events" in $\mathcal{E}$. Thus we need an algorithm (= operational definition) of what we mean by "length".

$\triangleleft$ Two event *types*:

$$\{e_L\} = \{\langle\text{Left end of rod detected}\rangle\} \tag{2.1a}$$

$$\{e_R\} = \{\langle\text{Right end of rod detected}\rangle\} \tag{2.1b}$$

Think of an event *type* as a set (equivalence class) of all elementary events that you deem ↑ *type-identical* (but not ↑ *token-identical*). In the example given here, there will be many events e_L in spacetime that signify "Left end of rod detected" (if there is one rod, there will be one such event for each time t); these are *different* events of the *same type* $\{e_L\}$.

One could even declare that the event *type* $\{e_L\}$ *is* what we refer to as "the left end of the rod."

→ Algorithm `LENGTH` to compute *"Length of Rod"* in system K at time t:

LENGTH:
➔ **Input:** Coincidences $\mathcal{E}$, Inertial system label K, Time t
← **Output:** Length l_K of rod at time t as measured in K

1. Find (unique) event $L \in \mathcal{E}$ with $\{e_L\} \in L$ and $(t, \vec{l})_K \in L$.
2. Find (unique) event $R \in \mathcal{E}$ with $\{e_R\} \in R$ and $(t, \vec{r})_K \in R$.
3. Return $l_K := |\vec{l} - \vec{r}|$.

Here, $\{e_L\} \in L$ is shorthand for $\{e_L\} \cap L \neq \emptyset$. In words: the coincidence class L contains an event of the *type* "Left end of rod detected".

Note that we define "length" as the spatial distance between the two ends of the rod *at the same time t* (as measured by the clocks in K). I hope you agree that this is what one typically means by "length."