↓Lecture 35 [09.08.24]

7 | Lorentz covariance:

Here we finally answer the question:

Why does the quantization of the bosonic string only work in $D = 26$ spacetime dimensions?

Note that so far there is no restriction on the normalization constant A and/or the spacetime dimension D (= number of scalar fields X^μ).

However, remember that we sacrificed manifest Lorentz covariance when fixing the light-cone gauge. (The return of this investment was a ghost-free quantum theory, i.e., a theory without negative-norm states in the Hilbert space; cf. ↑ *Covariant quantization*.) As our formulation is no longer manifestly Lorentz covariant, we cannot be sure that our *quantum* theory is still relativistic (that is, carries a representation of the Poincaré group).

If there is no representation of the Poincaré group on the Hilbert space of a given quantum theory, it is not relativistic. Remember that representations of symmetry groups are exactly that: they *represent* physical actions in the real world (translations, rotations, boosts, …) by linear operators on an abstract, mathematical state space (the Hilbert space). The defining feature of a *relativistic* quantum theory is that it specifies how e.g. a boost modifies the quantum state that describes your system, and that the combination of such transformations yields a multiplicative structure called "Poincaré group." (Note that the Hamiltonian of the theory is part of this representation as it is the generator of translations in time.)

So let us manually check whether the Hilbert space of the quantized (open) string is a representation of the Poincaré group:

i | Remember: Lie algebra of Lorentz group Eq. (4.69):
(➔ Problemset 5 of SPECIAL RELATIVITY course)

$$\left[J^{\mu\nu}, J^{\rho\sigma}\right] = \eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\nu\sigma} J^{\mu\rho} + \eta^{\mu\sigma} J^{\nu\rho}\,. \tag{15.107}$$

Here on a D-dimensional spacetime: $\mu, \nu = 0, 1, \dots, D-1$. Note that $J^{\mu\nu}$ are abstract elements of a Lie algebra, not operators.

$\triangleleft$ In particular the generator:

$$J^{-i} = \frac{1}{\sqrt{2}}\left(J^{0,i} - J^{D-1,i}\right) \tag{15.108}$$

If something bad happens to Lorentz symmetry, it most likely is related to this generator because X^- is a rather non-trivial function of the dynamical fields X^i via Eqs. (15.89) and (15.90).

Eq. (15.107) implies $\xrightarrow{\circ}$

$$\left[J^{-i}, J^{-j}\right] = 0 \tag{15.109}$$

The question is now whether there are *operators* J^{i-} (= representations) acting on the Hilbert space of the quantized string that satisfy this commutation relation. If not, we lost Lorentz symmetry and have a problem …

ii | Noether charges Eq. (15.66) $\xrightarrow{\text{Eq. (15.99)}}$ Representations of Lorentz group (?):

The very fact that the Witt algebra got spoiled by quantization should make us wary; after all, the Lorentz algebra might be affected by an anomaly as well!

(For once we mark these operators with a hat ˆ to mark them as *representations*.)

$$\hat{J}^{\mu\nu} \overset{15.66}{:=} T \int_0^\pi d\sigma \, \underbrace{:\!(X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu)\!:}_{\text{Normal ordered}}$$

$$\stackrel{\circ}{=} \underbrace{x^\mu p^\nu - x^\nu p^\mu}_{\text{Orbital angular momentum}} \underbrace{-i \sum_{n=1}^\infty \frac{1}{n} (\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n)}_{\text{Internal angular momentum (spin)}} \tag{15.110}$$

- $\hat{J}^{\mu\nu}$ is an operator on the light-cone Hilbert space $\mathcal{H}$ spanned by the transversal modes (→ Section 15.2.2). This Hilbert space must be a representation of the Lorentz group, i.e., the commutator algebra must be Eq. (15.107) and Eq. (15.109) must hold.
- The fact that $\hat{J}^{\mu\nu}$ has a contribution that can be interpreted as internal angular momentum (= spin) already suggests that different excitations of the string describe particles not only with different *masses* but also with different *spin*.
- Eq. (15.110) is a *definition* of the quantization of the classical charge $J^{\mu\nu}$. Due to the occurrence of quadratic terms in oscillator modes, it suffers from an ordering ambiguity similar to the Virasoro mode $L_0^\perp$. The operator is then again defined via ← *normal ordering*, such that the ground state/vacuum is Lorentz invariant. Note that the second summand in Eq. (15.110) is indeed normal ordered since the modes α^μ_{-n} (α^μ_n) are creation (annihilation) operators [← Eq. (15.100)].
- A global, continuous symmetry gives rise to a classically conserved quantity via Noether's theorem (for example: spatial translation symmetry leads to conservation of the total momentum). Quantizing the theory makes this quantity into an operator (for example: the momentum operator). In the absence of quantum anomalies, this operator is the generator of the original symmetry transformation (for example: the momentum operator generates spatial translations).

iii | In particular, we must set for the crucial operator Eq. (15.108):

$$\hat{J}^{-i} := x^- p^i - \underbrace{\tfrac{1}{2}\left(x^i p^- + p^- x^i\right)}_{\text{Symmetrized} \to \text{Hermitian}} - i \sum_{n=1}^\infty \frac{1}{n}\left(\alpha^-_{-n}\alpha^i_n - \alpha^i_{-n}\alpha^-_n\right) \tag{15.111a}$$

$$\overset{15.104}{\underset{15.90}{=}} x^- p^i - \frac{1}{4\alpha' p^+}\left[x^i (L_0^\perp - A) + (L_0^\perp - A) x^i\right] - \frac{i}{\sqrt{2\alpha'} p^+} \sum_{n=1}^\infty \frac{1}{n}\left(L^\perp_{-n}\alpha^i_n - \alpha^i_{-n} L^\perp_n\right) \tag{15.111b}$$

- Due to p^-, the Virasoro operator $L_0^\perp$ enters the stage. To account for its ordering ambiguity, we must augment the expression by the (yet undetermined) normal ordering constant A.
- Note that the generators of a symmetry group should also be *Hermitian*, such that the representation of the group is *unitary* (symmetries must not change the absolute values of overlaps of state vectors; ↑ *Wigner's theorem*). The mode term in Eq. (15.110) is

clearly Hermitian since $\alpha_n^{\mu\dagger} = \alpha_{-n}^{\mu}$. However, without a careful ordering of operators, this is not true for the orbital angular momentum term: since x^i does not commute with p^- [due to Eqs. (15.90) and (15.104)], the second term must be symmetrized. For details: ↑ Zwiebach [7] (§12.5, pp. 260–261).

iv | Plug Eq. (15.111b) in Eq. (15.109):

From Eq. (15.111b) we should expect the normal ordering constant A to show up. The commutator of Virasoro operators also certainly plays a role, so that the Virasoro algebra Eq. (15.106) with its central charge $c = D - 2$ enters the computation. It is therefore not surprising that the result depends on A and D:

$$\left[\hat{J}^{-i}, \hat{J}^{-j}\right] \overset{15.111b}{\underset{15.106}{=}} \underbrace{-\frac{1}{\alpha'(p^+)^2}\sum_{m=1}^{\infty}\Delta_m\left(\alpha_{-m}^i\alpha_m^j - \alpha_{-m}^j\alpha_m^i\right)}_{\neq 0 \,\to\, \uparrow \textit{Quantum anomaly} ☹} \tag{15.112}$$

with anomaly factors

$$\Delta_m = m\left(\frac{26-D}{24}\right) + \frac{1}{m}\left[\frac{D-26}{24} + (1-A)\right]. \tag{15.113}$$

So like the Witt algebra, the Lorentz algebra suffers from a *quantum anomaly*: the quantization modifies the algebra. Thus, whatever we quantized, it is no longer a relativistic string ☹ .

Except ...

→ Lorentz symmetry is broken *unless* $\forall m \in \mathbb{N}: \ \Delta_m = 0 \ \Leftrightarrow$

$$\boxed{A = 1 \quad \text{and} \quad D = 26} \tag{15.114}$$

- This result states that a relativistic string propagating on Minkowski space can only be quantized consistently in $D = 26$ spacetime dimensions. The constant $A = 1$ has consequences of similar importance for the masses of the particles predicted by the theory (→ Section 15.2.2).
- There are two perspectives on this result:
  - String-theory advocate: ☺☺☺

    Wow! String theory doesn't leave us any choice – it predicts *spacetime to be $D = 26$ dimensional (or $D = 10$ dimensional in superstring theory).*
  - String-theory opponent: ☹

    Bullshit! Our spacetime is not $D = 26$ but $D = 4$-dimensional. This theory cannot describe reality; it is a mathematical peculiarity, nothing more.

    Unfortunately, as just shown, we cannot simply "tweak" the theory to match $D = 4$; the quantum version of the relativistic string is *rigid*: $D = 26$ or we're out of business! A creative cop-out is to keep and "hide" the unwanted 22 spatial dimensions by curling them up into tiny circles (or more complicate ↑ *Calabi-Yau manifolds*; ↑ *compactification*). This modification of course affects the dynamics and interaction of strings propagating in the "large" four dimensions of our spacetime. How exactly the string dynamics is modified depends on how exactly one curls up the additional dimensions. Unfortunately, there are many different ways to do this ("unfortunately" is a word needed quite often in string theory); this leads to the ↑ *string theory landscape*, the ↑ *anthropic principle* raising its ugly head, and, eventually, the demise of the scientific principle …

- In ↑ *covariant (canonical) quantization*, Lorentz covariance is manifest throughout the computation (all operators have Lorentz indices), but for $D \neq 26$ the constructed representation is not unitary (= there are ghosts [negative-norm states] in the physical state space). By contrast, in light-cone quantization there are only positive-norm states in the Hilbert space, but for $D \neq 26$ the operators $J^{\mu\nu}$ no longer satisfy the Lorentz algebra and Lorentz covariance is lost.

15.2.2. Bosonic string spectrum

Now that we quantized the (open) bosonic string, we can start to build its Hilbert space [= the representation of the commutator algebra Eq. (15.99)]. The (now quantized) excitations of the string are identified with elementary particles of various masses and spins. Finding the Hilbert space is straightforward since Eq. (15.99) consists of canonical position and momentum operators, together with (multiple copies) of the harmonic oscillator algebra, both of which you studied in your first course on quantum mechanics.

We start with the open string:

8 | ⊲ Open string:

i | Eq. (15.99) → Canonical pairs (x^-, p^+) and (x^i, p^i)

→ Momentum space representation:

$$|k^+, \vec{k}_\perp\rangle \equiv |k\rangle \qquad \text{⁂ \textit{String ground states}} \tag{15.115}$$

with

$$\forall_{m\geq 1}: \; a^i_m |k\rangle = 0 \quad \text{and} \quad (p^+, \vec{p}_\perp)|k\rangle = (k^+, \vec{k}_\perp)|k\rangle \tag{15.116}$$

for $i = 1, \ldots, D-2$.

These states describe a single string in its oscillatory ground state with momentum $(k^+, \vec{k}_\perp)$.

ii | We can now create excitations of the string by acting with mode creation operators $a^{i\dagger}_n$ on these ground states:

→ Fock space $\mathcal{H}_o$ spanned by open string states

$$|\boldsymbol{\lambda}, k\rangle := \underbrace{\prod_{n=1}^{\infty}}_{\substack{\text{Oscillation}\\ \text{Mode}}} \underbrace{\prod_{i=1}^{24}}_{\substack{\text{Transversal}\\ \text{direction}}} \left(a^{i\dagger}_n\right)^{\lambda_{n,i}} |k\rangle \qquad \text{with } \lambda_{n,i} \in \mathbb{N}_0\,. \tag{15.117}$$

→ $|\boldsymbol{\lambda}, k\rangle$ = State of particle with mass squared (remember that $A = 1$ is now fixed)

$$M^2 |\boldsymbol{\lambda}, k\rangle \overset{15.105}{\underset{15.102}{=}} \frac{1}{\alpha'}(N^\perp_{\boldsymbol{\lambda}} - 1)|\boldsymbol{\lambda}, k\rangle \quad \text{with} \quad N^\perp_{\boldsymbol{\lambda}} = \sum_{n=1}^{\infty}\sum_{i=1}^{24} n\lambda_{n,i}\,. \tag{15.118}$$

- The state $|\boldsymbol{\lambda}=\mathbf{0},k=0\rangle$ describes a string with zero momentum and no oscillations, not the vacuum ("no string").
- Since there are infinitely many levels $N^\perp_{\boldsymbol{\lambda}}=0,1,2,\dots$ (and the irreducible representations of the Poincaré group live within these levels), string theory predicts *infinitely many particles*!
- The operators $J^{\mu\nu}$ generate a representation of the Lorentz group on the Hilbert space spanned by the states Eq. (15.117). This representation decomposes into a direct sum of ↓ *irreducible representations* of states that mix under Lorentz transformations. Since $M^2=-p^2\propto(N^\perp-1)$ is a Lorentz scalar, only states of the same level $N^\perp$ can transform into each other under Lorentz transformations.

  According to ↑ *Wigner's classification*, physical *particles* correspond to irreducible representations of the Poincaré group. (A particle type is the linear subspace of quantum states that can be transformed into each other by Poincaré transformations. This is why we say that an electron with spin up and an electron with spin down are the same type of particle: If you rotate your experiment, you can make one into the other.)

  This means that we should identify particles by the linear subspaces within each level of string the Hilbert space that are invariant under Lorentz transformations. Due to the light-cone gauge, this is a non-trivial task: We have only transversal modes $a_n^{i\dagger}$, but know that *massive* particles need more. These are provided by other modes in the same level, but the identification of the irreducible representations is rather involved for massive particles.

We can now study the particles that arise from the lowest levels of the string spectrum:

iii | ⊲ Lowest-mass excitations:

a | Level $N^\perp=0$: (this is the particle that corresponds to the string ground state)

$$|k\rangle\quad\text{with mass}\quad M^2=-\frac{1}{\alpha'}<0\quad\to\quad\text{⁂ } \textit{Tachyon}\text{ (Scalar)}\quad\to\quad\text{Unstable vacuum ☹}$$

- Since there is only one state $|k\rangle$ for each momentum in the lowest level, the particle must be a *scalar* (no internal degrees of freedom).
- Particles with $M^2<0$ are called *tachyons*. They have a space-like 4-momentum ($p^2=-M^2>0$) and therefore "move faster than the speed of light." This, however, is a misleading interpretation in the context of quantum field theories, where they are symptoms of an *unstable vacuum state* [64, 65] (← Section 4.4). The existence of this state makes bosonic string theory unstable and motivates its extension by fermions and supersymmetry; ↑ *superstring theory*.

  To understand the effect of negative square masses in a scalar field theory (here: the quantum field of the tachyons), recall that the generic ϕ^4-Lagrangian that governs such fields has the form

  $$\mathcal{L}=\tfrac{1}{2}(\partial\phi)^2-m^2\phi^2-\lambda\phi^4\equiv\tfrac{1}{2}(\partial\phi)^2-V(\phi)\,,\tag{15.119}$$

  with (bare) particle mass m and interaction λ; the potential is $V(\phi)=m^2\phi^2+\lambda\phi^4$. For $\lambda=0$, the classical EOM is the Klein-Gordon equation: $(\partial^2+m^2)\phi=0$. Note that for a lower-bounded potential energy it must be $\lambda>0$.

  For *positive* square mass $m^2>0$, the classical ground state is clearly $\phi=0=$ const. However, for *negative* square mass $m^2<0$ (now ϕ is a tachyon field!), the lowest potential energy is found for $|\phi| = \sqrt{-m^2/2\lambda} \neq 0$. This indicates ↓ *spontaneous symmetry breaking*, and therefore an *instability* of the (tachyon-)vacuum $\phi = 0$. What then happens is that the system transitions into the new vacuum by "condensing tachyons"; this vacuum has new excitations which have *positive* square mass (the Higgs mode). This is exactly what happens with the Higgs field in the Standard Model: The Higgs symmetry breaking can be understood as "tachyon condensation", and the excitations of the new (symmetry-broken) vacuum are the Higgs bosons.

The bottom line is that tachyons in the spectrum of a quantum field theory are not superluminal "science-fiction particles" but harbingers of a vacuum decay.

b | Level $N^\perp = 1$:

This level can only be reached by applying a single creation operator of the $n = 1$-mode:

$$\overbrace{|i,k\rangle \equiv a_1^{i\dagger}|k\rangle}^{D-2=24\text{ states}} \quad \text{with mass} \quad M^2 = 0 \quad \rightarrow \quad \text{Massless vector boson} \quad \rightarrow \quad \text{Photon} ☺$$

- Since $i = 1, \cdots, 24$ these states transform under the vector representation of SO(24), as one would expect for massless vector bosons in $D = 26$ dimensions. This allows us to identify them with the *photons* of electrodynamics in $D = 26$ dimensions:

$$\underbrace{\underbrace{|i,k\rangle = a_1^{i\dagger}|k\rangle}_{\text{String states @ } N^\perp = 1} \quad \leftrightarrow \quad \underbrace{a_k^{i\dagger}|0\rangle}_{\text{Photon states in } D=26}}_{\text{Same Poincaré representation \& momenta \& mass}} \tag{15.120}$$

Remember that in $D = 4$-dimensional electrodynamics there are $D - 2 = 2$ transverse polarizations for massless photons. These form helicity representations of SO(2). Analogously, the $D - 2 = 24$ transverse polarizations above form a representation of SO(24), the symmetry group of photons in $D = 24$ spacetime dimensions.

- The fact that there are $D - 2$ states on the first level is independent of the normal ordering constant $A = 1$. The latter, however, makes these states *massless* ($M^2 = 0$) and thereby consistent with the representation theory of the Poincaré group: $D-2$ states can form a vector representation of SO$(D-2)$ – but not of SO$(D-1)$. The latter is a subgroup of the Lorentz group SO$(1, D-1)$ and would be needed for *massive* particles. By contrast, *massless* particles transform exactly under SO$(D-2)$ (because you can only rotate them about their momentum vector).

  Fun fact: By this line of arguments you can actually infer $A = 1$ from the requirement of Lorentz covariance even before you know that $D = 26$.

c | Level $N^\perp > 1$: Massive bosonic particles …

For example, $N^\perp = 2$ can be reached by either applying $a_1^{i\dagger}a_1^{j\dagger}$ or $a_2^{i\dagger}$ to the string ground state; the mass of these particles is $M^2 = 1/\alpha' > 0$. In total there are $\frac{1}{2}(D-2)(D+1)$ states on level $N^\perp = 2$ (324 states for $D = 26$). These states form the representations of a so called ↑ *massive tensor boson*.

9 | ⊲ Closed string:

We start with the generalization of the results for the quantized open string to the closed string (without derivations). We can then study the closed string spectrum in analogy to the open string spectrum:

i | Quantized closed string in light-cone gauge:

a | Eq. (15.54) → Operator algebra for *closed* string:

$$\left[x^i, p^j\right] = i\delta^{ij} \tag{15.121a}$$

$$\left[p^+, x^-\right] = i \tag{15.121b}$$

$$\left[\alpha^i_m, \alpha^j_n\right] = m\delta_{m+n}\delta^{ij} \tag{15.121c}$$

$$\text{New!} \Rightarrow \quad \left[\tilde\alpha^i_m, \tilde\alpha^j_n\right] = m\delta_{m+n}\delta^{ij} \tag{15.121d}$$

- All commutators not shown vanish, of course.
- The only difference to the open string is that there is another independent copy Eq. (15.121d) of oscillator modes.
- According to Eq. (15.50), the interpretation of the two mode sets is that they create right- and left-moving oscillations on the string, respectively.

b | The only subtlety concerning the closed string is that there is a constraint connecting the left- and right-moving zero-modes:

$$\alpha_0^\mu \overset{\text{def}}{=} \sqrt{\frac{\alpha'}{2}}\,p^\mu \overset{\text{def}}{=} \tilde\alpha_0^\mu \quad\Rightarrow\quad \alpha_0^- = \tilde\alpha_0^- \quad\Rightarrow\quad L_0^\perp = \tilde L_0^\perp \tag{15.122}$$

- While α_n^μ and $\tilde\alpha_n^\mu$ are *independent* oscillator modes for $n \neq 0$, they are *the same* center-of-mass mode for $n = 0$ (there is only one string with one center-of-mass!).
- The analog of Eq. (15.104) for the closed string reads

$$\sqrt{2\alpha'}\alpha_0^- = \frac{2}{p^+}(L_0^\perp - 1) \quad\text{and}\quad \sqrt{2\alpha'}\tilde\alpha_0^- = \frac{2}{p^+}(\tilde L_0^\perp - 1)\,, \tag{15.123}$$

  i.e. the normal-ordering constants $A = 1$ and $\tilde A = 1$ required for Lorentz covariance are the same for left- and right-movers. (The critical dimension $D = 26$ is also the same.)

With the analog of Eq. (15.102)

$$L_0^\perp = \frac{\alpha'}{4}p_\perp^2 + N^\perp \quad\text{and}\quad \tilde L_0^\perp = \frac{\alpha'}{4}p_\perp^2 + \tilde N^\perp \tag{15.124}$$

this constraint translates into the …

→

$$N^\perp = \tilde N^\perp \qquad ⁂\ \textit{Level-matching condition} \tag{15.125}$$

with level operators

$$N^\perp := \sum_{n=1}^\infty n\, a_n^{i\dagger} a_n^i \quad\text{and}\quad \tilde N^\perp := \sum_{n=1}^\infty n\, \tilde a_n^{i\dagger} \tilde a_n^i\,. \tag{15.126}$$

→

The level-matching condition Eq. (15.125) excludes some states from the Fock space generated by the oscillator modes $a_n^{i\dagger}$ and $\tilde{a}_n^{i\dagger}$; i.e., only a subspace of the Hilbert space contains *physical states*.

c | The mass shell condition follows in analogy to Eq. (15.105):

$$M^2 = \frac{2}{\alpha'}\left(N^\perp + \tilde{N}^\perp - 2\right) \qquad \text{⁂ Mass shell condition}$$

- Combine Eqs. (15.122) to (15.124) to show this.
- The -2 differs from the open string case Eq. (15.118) since the normal ordering constant contributes twice in the sum (for both left- and right-moving modes).

ii | Fock space:

The closed string Fock space $\mathcal{H}_c$ is spanned by the states

$$|\boldsymbol{\lambda},\tilde{\boldsymbol{\lambda}},k\rangle := \underbrace{\left[\prod_{n=1}^{\infty}\prod_{i=1}^{24}\left(a_n^{i\dagger}\right)^{\lambda_{n,i}}\right]}_{\text{Right-moving modes}} \times \underbrace{\left[\prod_{m=1}^{\infty}\prod_{j=1}^{24}\left(\tilde{a}_m^{j\dagger}\right)^{\tilde{\lambda}_{m,j}}\right]}_{\text{Left-moving modes}} |k\rangle \tag{15.127}$$

Here again $\lambda_{n,i}, \tilde{\lambda}_{n,i} \in \mathbb{N}_0$.

if and only if they satisfy the level-matching condition Eq. (15.125):

$$\sum_{n=1}^{\infty}\sum_{i=1}^{24} n\lambda_{n,i} =: N_{\boldsymbol{\lambda}}^\perp \stackrel{!}{=} \tilde{N}_{\boldsymbol{\lambda}}^\perp := \sum_{n=1}^{\infty}\sum_{i=1}^{24} n\tilde{\lambda}_{n,i}\,. \tag{15.128}$$

This equation cannot be solved explicitly, it must be studied on a case-by-case basis!

iii | ◁ Lowest-mass excitations:

a | Level $N^\perp = \tilde{N}^\perp = 0$:

$$|k\rangle \quad \text{with mass} \quad M^2 = -\tfrac{4}{\alpha'} < 0 \quad \rightarrow \quad \text{Another} \leftarrow \textit{Tachyon} \text{ (Scalar)}$$

- The closed string tachyons are mathematically analogous to the open string tachyons.
- Closed string tachyons are less well understood than open string tachonys. ZWIEBACH [7] writes (§13.3, p. 291):

  > *The closed string tachyon is far less understood than the open string tachyon. […] The instabilities associated with closed string tachyons are expected to be instabilities of spacetime itself. They remain largely mysterious.*

  Congratulations! You have officially entered physics mystery land …

b | Level $N^\perp = \tilde{N}^\perp = 1$:

Such states must have always one left- and one right-moving mode with $n = 1$ excited:

$$|\Psi, k\rangle := \sum_{i,j} R_{ij}\, a_1^{i\dagger} \tilde{a}_1^{j\dagger} |k\rangle \quad \text{with} \quad M^2 |\Psi, k\rangle = 0 \text{ (massless)} \tag{15.129}$$

R_{ij}: arbitrary $(D-2) \times (D-2)$ square matrix

→ Decompose this matrix *w.l.o.g.* as follows:

$$\underbrace{R_{ij}}_{(D-2)^2} = \underbrace{\overbrace{S_{ij}}^{\substack{\text{Symmetric}\\ \text{\& traceless}}}}_{\frac{1}{2}(D-2)(D-1)-1} + \underbrace{\overbrace{A_{ij}}^{\text{Antisymmetric}}}_{\frac{1}{2}(D-2)(D-3)} + \underbrace{\overbrace{D\,\delta_{ij}}^{\text{Scalar} \times \text{Identity}}}_{1} \tag{15.130}$$

The expressions below the matrix components denote the number of degrees of freedom.

These three parts transform each as an irreducible representation under Poincaré transformations, i.e., these states correspond to different types of massless particles:

→ *Three* massless particle types:

$$|S, k\rangle := \textstyle\sum_{i,j} S_{ij}\, a_1^{i\dagger} \tilde{a}_1^{j\dagger} |k\rangle \qquad \text{Graviton states ☺} \tag{15.131a}$$

$$|A, k\rangle := \textstyle\sum_{i,j} A_{ij}\, a_1^{i\dagger} \tilde{a}_1^{j\dagger} |k\rangle \qquad \text{⁂ \textit{Kalb-Ramond} states ☹} \tag{15.131b}$$

$$|D, k\rangle := \textstyle\sum_{i} D\, a_1^{i\dagger} \tilde{a}_1^{i\dagger} |k\rangle \qquad \text{⁂ \textit{Dilaton} state ☹} \tag{15.131c}$$

- Remember from our discussion of gravitational waves (Section 13.4) that classical excitations of the metric field can be parametrized (after exploiting the gauge symmetry of GENERAL RELATIVITY) by a *symmetric*, *traceless* field $h_{\mu\nu}$ with only *transverse* modes. In D spacetime dimensions, this means that a gravitational wave can be encoded in a $(D-2) \times (D-2)$ matrix h_{ij} that is symmetric and traceless; it has therefore

  $$\tfrac{1}{2}(D-2)(D-1) - 1 = \tfrac{1}{2}D(D-3) \tag{15.132}$$

  physical degrees of freedom; in $D = 4$ one finds the *two* degrees of freedom that we identified in Section 13.4 as two possible polarizations.

  But Eq. (15.130) shows that there are exactly as many states $|S, k\rangle$ as demanded by Eq. (15.132), which tells us that these are the required massless "spin-2" states needed for a graviton. (Note that it is not obvious that this field couples to the other particles of the theory via the energy-momentum tensor.)

  For details: ↑ ZWIEBACH [7] (§10.6, pp. 209–212).

- The Kalb-Ramond states are the excitations of an antisymmetric, massless tensor field $B_{\mu\nu}$ that is similar to the electromagnetic gauge potential A_μ. It acts as a source of ← *torsion* for the affine connection on spacetime [← Eq. (10.41)].

- The dilaton states are the excitations of a massless scalar field, the ↑ *dilaton field*. Intuitively, the dilaton is the quantization of the "breathing" mode mentioned ← *earlier*. The dilaton modifies the theory of gravity predicted by string theory to a tensor-scalar type (← Section 12.3); it also controls the *interaction strength* of strings.

  For details: ↑ ZWIEBACH [7] (§13.4, pp. 294–296).

15.3. ‡ Closing remarks

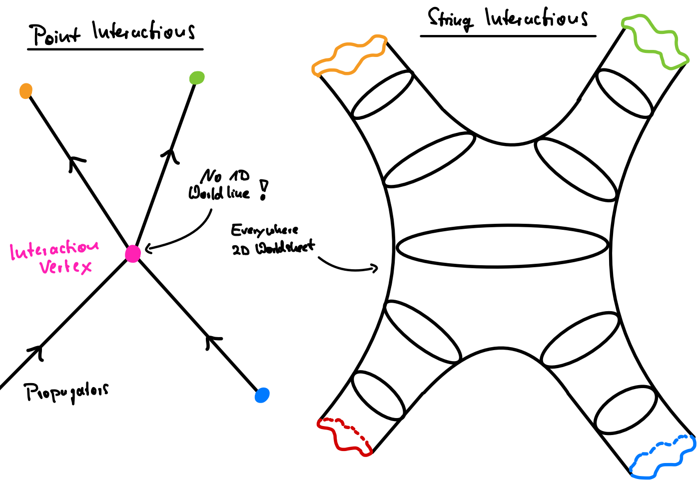
We conclude our excursion with a few comments on advanced topics:

For more details the reader is referred to David Tong's script on String Theory [319].

- *How do strings interact?*

  So far we described the states of a single string (open or closed) propagating on a $D = 26$-dimensional spacetime.

  1 | $\triangleleft$ Difference between theories of *point particles* and *strings*:



  - The action of a free point particle lives on a 1D worldline – and is therefore *undefined* on a vertex where two particles meet (the crossing of two lines is not a manifold).

    → Interaction terms must be added by hand.
  - The action of a free string lives on a 2D worldsheet and is well-defined everywhere on the worldsheet of two strings that merge and separate again.

    → The Polyakov action is already well-defined for interacting strings.

  2 | Given the gauge symmetries of the Polyakov action (= diffeomorphism invariance & Weyl symmetry), are there possible terms that we could add to modify string interactions?

  $\overset{\circ}{\rightarrow}$ Polyakov action can be extended by only one term:

$$\tilde{S}_P := -\int \mathrm{d}\tau \mathrm{d}\sigma \sqrt{h}\left[\frac{T}{2}h^{ab}\partial_a X^\mu \partial_b X_\mu + \overbrace{\frac{\lambda}{4\pi}R(h)}^{\text{Weyl \& Diff. invariant}}\right] \tag{15.133a}$$

$$\equiv S_P[h,X] - \lambda\chi[h] \tag{15.133b}$$

  $R(h)$: Ricci scalar *on worldsheet* (This is not the curvature of spacetime!)

  The general covariance of the new term is obvious. It's Weyl invariance (up to a total derivative!) can be checked by a straightforward calculation.

  3 | Interpretation:

  $\triangleleft$ Worldsheet without boundaries $\overset{\circ}{\rightarrow}$

$$\chi[h] = \frac{1}{4\pi}\int \mathrm{d}\tau \mathrm{d}\sigma \sqrt{h}R(h) \overset{\text{Gauss Bonnet}}{=} 2(1-g) \in \mathbb{Z} \tag{15.134}$$

g: Number of "handles" of the worldsheet (g is called the ↑ *genus* of the worldsheet.)

→ χ depends only on the *topology* of the worldsheet.

I.e., the action χ is invariant under *geometrical* deformations of the worldsheet.

4 | Superposition principle → Sum over all worldsheet *topologies* and *geometries* …

… consistent with the string scattering process under consideration:

$$\sum_{\substack{\text{Topologies}\\\text{Metrics}}} e^{i\tilde{S}_P} \sim \sum_{\text{Topologies}} \underbrace{e^{2i\lambda(g-1)}}_{\equiv g_s^{2i(g-1)}} \int_{\substack{\text{Fixed}\\\text{topology}}} \mathcal{D}X\,\mathcal{D}h\,e^{iS_P[h,X]} \tag{15.135}$$

$g_s = e^{\lambda}$: ⁂ *String coupling constant*

- String theory has therefore only two parameters: The string tension T and the string coupling constant g_s. These two numbers determine the scattering amplitudes of all particles predicted by the theory. This is of course a fascinating prospect: The Standard Model has ~ 18 free parameters (masses and coupling constants) that ask for an explanation. Unfortunately (☺), none of these parameters have been derived from (super-)string theory so far.
- String theory calculations typically make use of two perturbative expansions: one in α' (to capture interactions on the worldsheet) and one in g_s (summing over the number of "holes" in the worldsheet).

• *Where is* GENERAL RELATIVITY*? Where are the Einstein field equations?*

This was a course on SPECIAL RELATIVITY and GENERAL RELATIVITY, and our most precious result was the Einstein field Eq. (12.10) that controls the geometry of spacetime.

String theory claims to be a theory of quantum gravity; to earn this title, it should reproduce the Einstein field equations in some limit. Since string theory is formulated on a static background metric (→ *below*) – and not in a background-independent form – GENERAL RELATIVITY emerges in a rather esoteric way:

1 | Let us first generalize the Polyakov action to arbitrary cuved spacetimes:

$$\text{Eq. (15.22)} \xrightarrow{\eta_{\mu\nu}\mapsto g_{\mu\nu}(X)} S_g^{\text{flat}}[X] := -\frac{T}{2}\underbrace{\int g_{\mu\nu}(X)\partial_a X^{\mu}\partial^a X^{\nu}\,\mathrm{d}\sigma\,\mathrm{d}\tau}_{\uparrow\ \textit{Non-linear } \sigma\textit{-model}} \tag{15.136}$$

→ *Interacting* quantum field theory with infinitely many coupling constants $g_{\mu\nu}(x)$

This explains why used flat Minkowski space $\eta_{\mu\nu}$ to quantize the bosonic string.

Actions like Eq. (15.136) with fields that map into non-linear manifolds (here: curved spacetime) are called ↑ *non-linear* σ*-models*; a well-studied class of interacting quantum field theories that also have applications in condensed matter physics.

→ *Question:* Is $g_{\mu\nu}(x)$ arbitrary?

Surprising answer: Not if we want the conformal *gauge* symmetry to be unbroken!

2 | To show this ⊲ Small fluctuations of the fields around x_0^μ:

$$X^\mu(\tau,\sigma) \equiv x_0^\mu + \sqrt{\alpha'}Y^\mu(\tau,\sigma) \tag{15.137}$$

$$\Rightarrow \quad g_{\mu\nu}(X) \stackrel{\circ}{=} \eta_{\mu\nu} - \frac{\alpha'}{3} \underbrace{R_{\mu\alpha\nu\beta}(x_0)}_{\text{Riemann curvature tensor}} Y^\alpha Y^\beta + \mathcal{O}(Y^3) \tag{15.138}$$

Here we use ← *locally inertial coordinates* [← Eq. (10.89)] on spacetime to make the first derivatives of the metric vanish. One can show that the (non-vanishing) quadratic order is then given by the ← *Riemann curvature tensor* (↑ *Riemann normal coordinates*).

Eq. (15.136) → Interacting quantum field theory on 2D worldsheet:

$$S_g^{\text{flat}}[Y] \approx -\frac{1}{4\pi}\int \mathrm{d}\sigma\mathrm{d}\tau \Big[\underbrace{\eta_{\mu\nu}\partial_a Y^\mu \partial^a Y^\nu}_{\text{Free fields}} - \underbrace{\frac{\alpha'}{3}R_{\mu\alpha\nu\beta}Y^\alpha Y^\beta \partial_a Y^\mu \partial^a Y^\nu}_{\text{Interaction}}\Big] \tag{15.139}$$

One can now apply the well-honed machinery of quantum field theory to this action:

→ Feynman rules & Perturbation theory in α' ...

3 | Remember: The conformal symmetry Eq. (15.30) of the Polyakov action is a *gauge* symmetry.

→ Consistency requires that it remains *unbroken* after quantization.

This means that a ← *quantum anomaly* that spoils this symmetry cannot be tolerated.

→ How to check the conformal symmetry of Eq. (15.139) after quantization?

Idea: Calculate ↑ *renormalization flow* of couplings $g_{\mu\nu}$:

Conformal symmetry ⇔ Scale invariance ⇔ RG fixed point

- Note that conformal transformations include *global scale transformations*; scale invariance is therefore a necessary (and under most circumstances sufficient) condition for conformal symmetry.
- The idea of the ↑ *renormalization group (RG)* is to study how the couplings in a Lagrangian change if one "zooms" our. Mathematically, one integrates out a thin shell of high-energy momenta from the partition sum (in perturbation theory for interacting QFTs) and studies how this changes the coupling constants of the obtained effective action (here: $g_{\mu\nu}$). As a result, one obtains differential equations that encode the change of coupling constants with changing length/energy scale λ. This is called the ↑ *renormalization flow*, and the function that determines the flow of a coupling constant is called its ↑ *beta function*:

$$\beta_{\mu\nu}(g) := \lambda \frac{\partial G_{\mu\nu}(X;\lambda)}{\partial\lambda}\,. \tag{15.140}$$

  A vanishing beta functions means that the theory "looks the same" on all length scales, i.e., is *scale invariant*.

4 | Apply standard techniques to compute RG flow of Eq. (15.139) in first order of α':

$$\stackrel{*}{\rightarrow} \qquad \beta_{\mu\nu}(g) = \alpha' R_{\mu\nu} + \mathcal{O}(\alpha'^2) \tag{15.141}$$

$R_{\mu\nu}$: Ricci tensor of $g_{\mu\nu}$

The resulting RG flow is called the ↑ *Ricci flow*; it is an important concept for the RG analysis of non-linear σ-models in general.

5 | We can finally piece everything together:

$$\begin{aligned} \text{No conformal anomaly!} \quad &\Rightarrow \quad \beta_{\mu\nu}(g) \stackrel{!}{=} 0 \\ &\Rightarrow \quad R_{\mu\nu} \stackrel{!}{=} 0 \\ &\Rightarrow \quad \text{Vacuum Einstein field equations ☺} \\ &\Rightarrow \quad \text{GENERAL RELATIVITY ☺} \end{aligned}$$

- This means that the conformal anomaly only cancels in $D = 26$ spacetime dimensions and if the *spacetime metric satisfies the Einstein field equations*!
- Similar results can be obtained with matter fields, i.e., also the coupling to the energy-momentum tensor follows from the constraint of conformal invariance.
- Computing the beta function to higher orders in α' (= evaluate Feynman diagrams with more than one loop) yields *quantum corrections* to the Einstein field equations (as expected for a theory of quantum gravity).

- *How do the gravitons of string theory relate to the spacetime metric?*

  So far we only found massless states of the closed string that transform under the correct representation of the Poincaré group (that of a symmetric, traceless rank-2 tensor field). We called these states "gravitons" – but it is not clear that (and how) these states relate to the metric of spacetime (which enters string theory not as a dynamical field but as a static background).

  Here is a *sketch* (!) how one can establish this relation:

1 | ⊲ String scattering amplitude of $i = 1, \dots, N$ string states:

Scattering amplitudes are calculated from the path integral via the insertion of so called ↑ *Vertex operators*. Each in- and out-going string state corresponds to a particular vertex operator (↑ *Operator-state correspondence*):

$$\mathcal{M}(V_1, \dots, V_N) \sim \int \mathcal{D}X\,\mathcal{D}h\; e^{iS_g^{\text{flat}}[h,X]} \prod_{i=1}^N V_i[h, X] \tag{15.142}$$

V_i: ↑ *Vertex operators*

2 | One can show that the Vertex operator of *single graviton state* has the form:

$$|\boldsymbol{S}, k\rangle \quad \leftrightarrow \quad V_{\boldsymbol{S},k} \sim \int \mathrm{d}\sigma^2\, S_{\mu\nu}(\partial_a X^\mu)(\partial^a X^\nu)\, e^{ik_\mu X^\mu} \tag{15.143}$$

Here k^μ is the momentum of the graviton and $S^{\mu\nu}$ encodes its polarization; cf. Eq. (15.131a) in light-cone gauge.

3 | We are interested in the effect of *quantized* gravitons on the *classical* background metric. Hence we should study graviton states that are "as classical as possible" (= minimize uncertainty relations). These states are ↓ *coherent states* that describe superpositions of many graviton excitations; just like classical laser light is not described by single photon states but by coherent states of photons.

Remember that the coherent state of a bosonic mode (e.g., a photon mode) is obtained by exponentiating the creation & annihilation operators:

$$\text{Coherent state:} \quad |\alpha\rangle = e^{\alpha a^\dagger - \alpha^* a}|0\rangle \tag{15.144}$$

→ ⊲ Vertex operator for *Coherent state* of gravitons:

$$V_{\boldsymbol{S},k}^{\text{Coherent}} \sim e^{iV_{\boldsymbol{S},k}} \tag{15.145}$$

4 | We can now study how the presence of such a coherent state affects scattering amplitudes:

$\sphericalangle$ Scattering amplitude with coherent state:

$$\mathcal{M} \sim \int \mathcal{D}X\mathcal{D}h\, e^{iS_g^{\text{flat}}} V_{S,k}^{\text{Coherent}} \underbrace{\dots}_{\substack{\text{Other}\\ \text{vertex}\\ \text{operators}}} \tag{15.146}$$

Observation: The coherent graviton state has the same form as the Polyakov action:

$$e^{iS_g^{\text{flat}}} e^{iV_{S,k}} = \exp\left\{ i \int \mathrm{d}\sigma^2 \underbrace{\left[g_{\mu\nu}(X) + S_{\mu\nu} e^{ik_\mu X^\mu} \right]}_{=:\bar{g}_{\mu\nu}(X)} \partial_a X^\mu \partial^a X^\nu \right\} \tag{15.147}$$

$\bar{g}_{\mu\nu}(X)$: New background metric

→ This demonstrates that ...

Coherent graviton state ≡ Modification of classical background metric

In conclusion, one can think of the static background metric $g_{\mu\nu}$ as a "condensate" of gravitons around which the quantum theory is developed. Gravitons are then the quantum fluctuations on top of this condensate.

- *What about Superstring Theory?*

  Here we studied *bosonic* string theory: We only found particles with *integer* spin that commute when exchanged (↑ *Spin-statistics theorem*). Since our world very much relies on the existence of *fermions* with half-integer spin (electrons etc.), this begs the question:

  Where are the fermions?

  1 | Answer: They are put in by hand:

$$S_{\text{SS}}^{\text{flat}}[X,\Psi] := -\frac{1}{4\pi\alpha'} \int \mathrm{d}\tau\mathrm{d}\sigma \underbrace{\left[\underbrace{\partial_a X^\mu \partial^a X_\mu}_{\substack{\text{Polyakov}\\ \text{(Bosonic string)}}} + \underbrace{\alpha' \bar{\Psi}^\mu \gamma^a \partial_a \Psi_\mu}_{\text{Worldsheet fermions}} \right]}_{\uparrow\ \textit{Superstring}} \tag{15.148}$$

  $\Psi_\alpha^\mu(\tau,\sigma)$: ↑ *Majorana fermions*

  These are real-valued two-component ↑ *Grassmann fields*; i.e., $\alpha = 1,2$ and $\mu = 0,\dots,D-1$ and $\Psi_\alpha^\mu \Psi_\beta^\nu = -\Psi_\beta^\nu \Psi_\alpha^\mu$.

  - You shouldn't be worried about the vector index on the two-component spinors Ψ^μ; they play the same role as for the worldsheet *scalars* X^μ.
  - We interpreted the bosonic fields X^μ as the spacetime positions of the string. The worldsheet fermions Ψ^μ do not have such a natural interpretation. They describe internal fermionic degrees of freedom that propagate along the string itself.
  - The action Eq. (15.148) has a new continuous symmetry that mixes the bosonic fields X^μ with the fermionic Grassmann fields Ψ^μ; this symmetry is called ↑ *supersymmetry* and gives ↑ *Superstring Theory* its name. (Supersymmetry guarantees the absence of tachyons and is therefore needed for the theories consistency.)

- The Dirac adjoint is $\bar{\Psi}^\mu := \Psi^{\mu\dagger}\gamma^0$ and the *two* Dirac matrices (the worldsheet is two-dimensional!) can be chosen as

$$\gamma^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{acting on} \quad \Psi^\mu = \begin{pmatrix} \Psi_1^\mu \\ \Psi_2^\mu \end{pmatrix}; \tag{15.149}$$

  they satisfy $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$.
- While there are good mathematical reasons to add fermions to the string action (→ *next*), there is no physical intuition that underlies their existence. These fermionic degrees of freedom on the string are simply *postulated*, just like the existence of the string itself. Superstring theory therefore does note *explain* the existence of fermions (like some ↑ *topologically ordered* condensed matter systems are able to). If you don't find this satisfying, I would agree.

2 | Teaser:

Besides the emergence of fermionic particles (= spacetime fermions) in the spectrum of the superstring, this extended theory has additional advantages over the bosonic string:

$$\underbrace{\text{String theory + Supersymmetry}}_{\text{Superstring theory}} \quad \to \quad \begin{cases} \text{Critical dimension } D = 10 < 26 \text{ ☺☹} \\ \text{Fermions included ☺} \\ \text{No Tachyon ☺} \end{cases}$$

- The bosonic string (= Polyakov action) is essentially unique. This is not so for the superstring: There are *five* distinct ways to define consistent supersymmetric string theories: Type I, Type IIA, Type IIB, Heterotic $E_8 \times E_8$, and Heterotic SO(32). They are conjectured to be limits of a single theory dubbed ↑ *M-Theory*.

- Meta-Knowledge:

The calculations in this chapter were rather involved. It is important, however, to keep in mind that this is physics from the 1970s; this version of string theory is not representative for the sophistication of the field today. Here is a sketch to provide perspective: