↓Lecture 34 [08.08.24]

15.1.1. Light-cone gauge

We could quantize the string right away by replacing the Poisson brackets of the oscillator modes with commutators. However, our theory still has unfixed residual gauge degrees of freedom that lead to problems after quantization. Here we identify and fix these gauge degrees of freedom, and finally impose the constraint $T_{ab} \overset{!}{=} 0$. The result will be a classical formulation of the relativistic string that can be canonically quantized without any issues (almost ...):

14 | Remember: We are in Flat gauge Eq. (15.43): $h_{ab} = \eta_{ab} = \mathrm{diag}\,(-1,+1)_{ab}$

But remember also: Polyakov action has ← *conformal symmetry* Eq. (15.30)

Conformal symmetries are *residual gauge symmetries* on the world sheet that allow for the transformation of the fields X^μ without modifying the world sheet metric h_{ab}.

→ ∢ Combinations of diffeomorphisms & Weyl transformations consistent with flat gauge:

- ∢ *Infinitesimal* diffeomorphism: $\bar\sigma^a = \sigma^a + \varepsilon^a(\sigma)$ [← Eq. (11.90)]

$$\text{Eq. (11.103)} \to \quad \delta h_{ab}^{\text{Diff}} = -(\partial_b\varepsilon_a + \partial_a\varepsilon_b) - \underbrace{\varepsilon^c\partial_c\eta_{ab}}_{=0} \tag{15.76}$$

The signs are different from Eq. (11.103) because the tensor is covariant.

- ∢ *Infinitesimal* Weyl transformation (15.28): $(|\rho|\ll 1)$

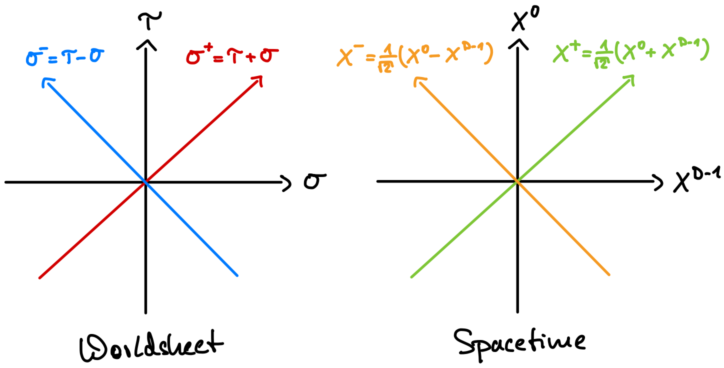
$$\tilde h_{ab} = e^{2\rho}\eta_{ab} \approx (1+2\rho)\eta_{ab} \quad\Rightarrow\quad \delta h_{ab}^{\text{Weyl}} = 2\rho\eta_{ab} \tag{15.77}$$

15 | → *Infinitesimal* conformal transformation:

$$\delta h_{ab}^{\text{Diff}} + \delta h_{ab}^{\text{Weyl}} \overset{!}{=} 0 \quad\Leftrightarrow\quad 2\rho\eta_{ab} = \partial_b\varepsilon_a + \partial_a\varepsilon_b \tag{15.78}$$

This differential equation must be solved for ε^a and ρ.

16 | To proceed, it is convenient to introduce new coordinates on the world sheet and on spacetime:



- On spacetime, introduce ⁂ *light-cone fields*:

$$X^\pm := \frac{1}{\sqrt{2}}(X^0 \pm X^{D-1}) \tag{15.79}$$

and analogously $p^\pm := \frac{1}{\sqrt{2}}(p^0 \pm p^{D-1})$ etc.

The choice of the space-like components X^{D-1} is arbitrary. Rewriting the theory in these variables singles out the direction $\mu = D-1$ and breaks manifest Lorentz covariance. This is the price we have to pay for a quantization without gauge-degrees of freedom (→ *below*).

- On the world sheet, define ⁂ *light-cone coordinates*:

$$\sigma^\pm := \tau \pm \sigma \quad \overset{\circ}{\Rightarrow} \quad \begin{cases} \partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma) \\ \Box = -4\partial_+\partial_- \\ \mathrm{d}s^2 = -\mathrm{d}\sigma^+\mathrm{d}\sigma^- \\ \begin{pmatrix} \eta_{++} & \eta_{+-} \\ \eta_{--} & \eta_{--} \end{pmatrix} = \begin{pmatrix} 0 & -1/2 \\ -1/2 & 0 \end{pmatrix} \\ \begin{pmatrix} \eta^{++} & \eta^{+-} \\ \eta^{--} & \eta^{--} \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} \end{cases} \tag{15.80}$$

Light-cone coordinates are simply inertial coordinates rotated by $\pm 45°$; i.e., both coordinate vectors point along null cones and are therefore light-like.

17 | Eq. (15.78) $\overset{\circ}{\rightarrow}$ Constraints on conformal transformations in light-cone coordinates:

$$\partial_-\varepsilon^- + \partial_+\varepsilon^+ = 2\rho \quad \Rightarrow \quad \rho = \rho(\varepsilon) \tag{15.81a}$$

$$\partial_+\varepsilon^- = 0 \quad \Rightarrow \quad \varepsilon^- = \varepsilon^-(\sigma^-) \tag{15.81b}$$

$$\partial_-\varepsilon^+ = 0 \quad \Rightarrow \quad \varepsilon^+ = \varepsilon^+(\sigma^+) \tag{15.81c}$$

→ Non-infinitesimal conformal transformation:

$$\bar{\sigma}^+ = \bar{\sigma}^+(\sigma^+) \quad \text{and} \quad \bar{\sigma}^+ = \bar{\sigma}^-(\sigma^-) \tag{15.82}$$

The above derivation shows that for each such diffeomorphism a Weyl transformation ρ exists to keep the metric in flat gauge ($h_{ab} = \eta_{ab}$). We do not need to known the specific form of ρ because we drop the conformal scaling factor anyway. So the point is that we can make any transformation of the form Eq. (15.82) *while keeping the flat gauge fixed*.

18 | Define a rescaled time coordinate:

$$\bar{\tau}(\sigma^+,\sigma^-) := \frac{1}{2}\left[\bar{\sigma}^+(\sigma^+) + \bar{\sigma}^-(\sigma^-)\right] \quad \Leftrightarrow \quad \Box\bar{\tau} = -4\partial_+\partial_-\bar{\tau} = 0 \tag{15.83}$$

Note that the two expressions are *equivalent*. But this implies that the only constraint on the new world sheet coordinate $\bar{\tau} = \bar{\tau}(\sigma^+,\sigma^-) = \bar{\tau}(\tau,\sigma)$ is that is satisfies the wave equation. Conversely, whenever we have a function on the world sheet that satisfies the wave equation, we can *w.l.o.g.* set it equal to (an affine function of) τ.

Compare this to the EOM (15.47) that the fields X^μ satisfy:

$$\Box X^\mu(\tau,\sigma) = 0 \quad \Rightarrow \quad \Box X^+(\tau,\sigma) = 0 \tag{15.84a}$$

$$\Rightarrow \quad X^+(\tau,\sigma) = X_R^+(\sigma^-) + X_L^+(\sigma^+) \tag{15.84b}$$

That we focus on the light-cone field X^+ is arbitrary; it becomes useful below.

19 | Thus we can always choose world sheet coordinates (τ, σ) (we omit bars) such that …

$$X^+(\tau,\sigma) = 2\alpha' p^+ \tau \qquad ⁂ \textit{ Light-cone gauge} \tag{15.85}$$

- In this gauge, the +-oscillator modes of the string are not excited and "frozen."
- That we pick out X^+ seems arbitrary at this point; it becomes useful → *below*. [We could have chosen any (linear combination of) X^μ to be an affine function of τ.]

20 | We are finally in the position to Enforce the constraint Eq. (15.62):

$$T_{ab} \overset{!}{=} 0 \quad \Leftrightarrow \quad (\dot X \pm X')^2 \overset{!}{=} 0 \tag{15.86}$$

Expand the contraction in the square in light-cone fields:

$$-2\underbrace{(\dot X \pm X')^+}_{\substack{=2\alpha' p^+ \\ \text{Light-cone gauge} ☺}} (\dot X \pm X')^- + \underbrace{\sum_{i=1}^{D-2} (\dot X \pm X')^i (\dot X \pm X')^i}_{\equiv (\dot X \pm X')^2_\perp} \overset{!}{=} 0 \tag{15.87}$$

We omit transversal sum symbols over $i = 1, \dots, D-2$ in the following.

→ The constraint can be satisfied by setting:

$$(\dot X \pm X')^- := \frac{1}{4\alpha' p^+} (\dot X \pm X')^2_\perp \tag{15.88}$$

→ Also the X^- degrees of freedom are no longer dynamically independent.

→ We must quantize only the $D-2$ *transversal* components X^i (and p^+).

21 | ∢ *Open* string (for simplicity, similar arguments hold for the closed string)

Recall that the mode expansion for the open string reads:

$$\text{Eq. (15.56)} \rightarrow \quad X^- = x^- + 2\alpha' p^- \tau + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{\alpha_n^-}{n} e^{-in\tau} \cos(n\sigma) \tag{15.89}$$

We use this (and the mode expansions for X^i) to express Eq. (15.88) in terms of modes:

Eq. (15.88) $\overset{\circ}{\rightarrow}$ (to show this set $\tau = 0$)

$$\sqrt{2\alpha'}\alpha_n^- = \frac{1}{p^+}\left(\frac{1}{2}\sum_{m=-\infty}^{\infty} \alpha_{n-m}^i \alpha_m^i\right) \equiv \frac{1}{p^+} L_n^\perp \tag{15.90}$$

(Note that this includes $\sqrt{2\alpha'} p^- = \alpha_0^-$ [for the open string].)

with ⁂ *Transversal Virasoro modes*:

$$L_n^\perp := \frac{1}{2}\sum_{m=-\infty}^{\infty} \alpha_{n-m}^i \alpha_m^i \tag{15.91}$$

That $\alpha_n^- \propto L_n^\perp$ is *quadratic* in the transversal modes α_n^i is evident from Eq. (15.88).

22 | To sum up, we have fixed the *flat gauge* and the *light-cone gauge*. In these gauges, the dynamics of the classical relativistic string is described by the following canonical pairs of variables:

$$\begin{aligned}
\text{Transversal modes:} &\quad \underbrace{X^i \text{ and } \Pi^i \; (i = 1,\dots,D-2)}_{\text{Equivalently: } x^i,\, p^i,\, \alpha^i_m \; (m\neq 0)} \\
\text{Light-cone position \& momentum:} &\quad x^- \text{ and } p^+
\end{aligned}$$

Note that $x^+ = 0$ is frozen [Eq. (15.85)] and $p^- \propto \alpha_0^-$ is determined via Eq. (15.90) and therefore also no longer dynamical.

These variables satisfy the Poisson algebra Eq. (15.58) for $i = 1, \cdots, D-2$:

$$\{x^i, p^j\} = \delta^{ij} \tag{15.92a}$$

$$\{p^+, x^-\} = 1 \tag{15.92b}$$

$$\{\alpha^i_m, \alpha^j_n\} = im\delta_{m+n}\delta^{ij} \tag{15.92c}$$

The generalization to the closed string is straightforward and will not be shown in detail.

With this we are ready to quantize the open string! But before we do this, one last thing …

23 | Witt algebra:

The Poisson algebra of the transversal oscillator modes determines the Poisson algebra of the transversal Virasoro modes (using the bilinearity and product rule for the Poisson bracket):

Eqs. (15.91) and (15.92) $\overset{\circ}{\rightarrow}$

$$\left\{L^\perp_m, L^\perp_n\right\} = i(m-n)L^\perp_{m+n} \qquad ⁂ \; \textit{Witt algebra} \tag{15.93}$$

- Canonical quantization is the prescription to replace classical Poisson brackets of phase space functions by the commutators of operators on a Hilbert space. However, this prescription is not well-defined for quadratic functions like the Virasoro modes. We will find that after quantization (and a suitable definition of quantized Virasoro *operators*) the Witt algebra will be *modified* by a ↑ *central extension*. This unexpected modification signifies a ↑ *quantum anomaly*, and is directly linked to the critical dimension $D = 26$ of bosonic string theory.
- The Witt algebra shows up due to the conformal symmetry Eq. (15.82) of the Polyakov action. That the Witt algebra (interpreted as an abstract Lie algebra) describes conformal transformations in two dimensions can be seen as follows:

  Remember that conformal transformations on (some region of) $\mathbb{R}^2 \simeq \mathbb{C}$ are given by ↓ *meromorphic functions* $f(z)$ on $\mathbb{C}$; these can be expanded in a Laurent series:

  $$\tilde{z} \equiv f(z) = z - \textstyle\sum_{n=-\infty}^{\infty} a_n z^n \,. \tag{15.94}$$

  An infinitesimal conformal transformation ($|a_n| \ll 1$) changes a scalar field $\phi(z)$ (for simplicity assumed to be holomorphic) as follows:

  $$\phi(z) \overset{\text{Scalar}}{=} \tilde{\phi}(\tilde{z}) \overset{\text{Taylor}}{\approx} \tilde{\phi}(z) - \left(\textstyle\sum_n a_n z^n\right) \partial_z \tilde{\phi}(z) \,. \tag{15.95}$$

Thus the generators of such transformations have the form

$$\delta_a \phi \equiv \tilde{\phi}(z) - \phi(z) \approx \left(\sum_n a_n z^n \partial_z\right) \phi(z) \equiv \left(\sum_n a_n L_{1-n}\right) \phi(z) \tag{15.96}$$

with generator basis $L_n := z^{1-n}\partial_z$. The Lie algebra of these generators is:

$$[L_m, L_n]\,\phi(z) = (z^{1-m}\partial_z)(z^{1-n}\partial_z)\phi(z) - (z^{1-n}\partial_z)(z^{1-m}\partial_z)\phi(z) \tag{15.97a}$$

$$= (m-n)z^{1-m-n}\partial_z\,\phi(z) \tag{15.97b}$$

$$= (m-n)L_{m+n}\,\phi(z)\,. \tag{15.97c}$$

That is, the Witt algebra is the Lie algebra of the "group" of conformal transformations. (The missing i can be obtained by redefining $L_m \mapsto -iL_m$.)

15.2. Quantization of the relativistic string

We quantize the string canonically, by replacing phase-space variables by operators and the Poisson algebra by a commutator algebra. The result will be a "first quantized" string, i.e., a relativistic quantum theory that describes a single string. Mathematically, this is achieved by techniques of "second quantization" because the string is described by a field theory.

There are three approaches to quantize the bosonic string:

- ↑ *Covariant canonical quantization*

  Pros: Manifestly Lorentz covariant | Cons: Unphysical states & ghosts (= negative norm states)

  This route starts by canonically quantizing Eq. (15.58) *without* fixing the light-cone gauge and enforcing the constraint Eq. (15.62) on the classical level. It is akin to ↑ *Gupta-Bleuler quantization* of the electromagnetic field.

- Light-cone quantization (→ Section 15.2.1)

  Pros: No unphysical states & ghosts | Cons: Not manifestly Lorentz covariant

  This is the approach taken below; it is akin to the quantization of the electromagnetic field in Coulomb gauge usually presented in courses on advanced quantum mechanics.

- ↑ *Covariant path integral quantization*

  This is the modern approach used in string theory (it is more abstract & versatile, but less suited for a first introduction).

  This approaches leverages the full machinery of quantum field theory and is akin to the ↑ *Faddeev-Popov quantization* of the electrommagnetic field [20].

Based on our preliminary work in Section 15.1 we can already conclude:

> The "first quantized" string is described by a *quantum field theory* of D scalar fields X^μ that live on the 1+1-dimensional world sheet.

There is also a "second" quantization of string theory: ↑ *string field theory*.

15.2.1. Light-cone quantization

¡! We focus again on the *open* string for simplicity and state results for the closed string → *later*.

1 | Remember: ↓ *Canonical quantization*:

$$\text{Poisson bracket}: \quad \{\bullet,\bullet\} \quad \rightarrow \quad \frac{1}{i\hbar}[\hat{\bullet},\hat{\bullet}] \quad : \text{Commutator} \tag{15.98}$$

In the following we set $\hbar = 1$ an omit hats ˆ for operators.

2 | Eq. (15.92) → Operator algebra for *open* string:

$$\left[x^i, p^j\right] = i\,\delta^{ij} \tag{15.99a}$$

$$\left[p^+, x^-\right] = i \tag{15.99b}$$

$$\left[\alpha^i_m, \alpha^j_n\right] = m\delta_{m+n}\delta^{ij} \tag{15.99c}$$

For the *closed* string, this algebra is extended by modes $\tilde{\alpha}^i_m$ in a straightforward way, cf. Eq. (15.54).

$\sphericalangle\, m > 0$ → Only non-vanishing commutator:

$$\underbrace{\left[\alpha^i_m, \alpha^j_{-m}\right] = m\delta^{ij}}_{\text{Harmonic oscillator?}} \rightarrow \left\{\begin{aligned} a^i_m &:= \frac{1}{\sqrt{m}}\alpha^i_m \\ a^{i\dagger}_m &:= \frac{1}{\sqrt{m}}\alpha^i_{-m}\end{aligned}\right\} \rightarrow \underbrace{\left[a^i_m, a^{j\dagger}_m\right] = \delta^{ij}}_{\text{Harmonic oscillator ☺}} \tag{15.100}$$

The excitations of an open string are thus described by a set of harmonic oscillator modes, labeled by the (transversal) direction $i = 1,\dots,D-2$ and mode $m = 1,2,\dots$ of the oscillation.

3 | Virasoro *modes* Eq. (15.91) $\xrightarrow{\text{Quantization}}$ Virasoro *operators*:

Problem: Ordering ambiguity for $L^\perp_0$:

$$L^\perp_{n\neq 0} = \frac{1}{2}\sum_{m=-\infty}^{\infty}\underbrace{\alpha^i_{n-m}\alpha^i_m}_{\text{Commute!}} \quad \text{but} \quad L^\perp_0 = \frac{1}{2}\sum_{m=-\infty}^{\infty}\underbrace{\alpha^i_{-m}\alpha^i_m}_{\text{Do not commute!}} \tag{15.101}$$

What is the correct ordering *for quantization?*

We do not know! So let us play it safe and not fix the ordering prematurely:

i | To this end, we first *define* the operator $L^\perp_0$ as ↑ *normal ordered*:

$$L^\perp_0 := \tfrac{1}{2}\alpha^i_0\alpha^i_0 + \sum_{n=1}^{\infty}\underbrace{\alpha^i_{-n}\alpha^i_n}_{\text{Normal ordered}} \overset{15.100}{=} \alpha'\underbrace{p^i p^i}_{=:\,p^2_\perp} + \underbrace{\sum_{n=1}^{\infty} n\, a^{i\dagger}_n a^i_n}_{=:\,N^\perp} \tag{15.102}$$

$N^\perp$: Transverse ⁂ *level operator*

Normal ordering is a prescription (a ↑ *meta operator*) to order strings of non-commuting creation- and annihilation operators such that creation operators are on the left and annihilation operators on the right (this ordering is done without commutation relations). The result is an operator with vanishing expectation value wrt. the vacuum/ground state $|0\rangle$. Normal ordering is often indicated by enclosing an expression by colons: $:\bullet:$.

ii | But we do not know the correct ordering for quantization! Conveniently, *all possible* orderings can be brought into the normal ordered form Eq. (15.102) by using the commutator algebra Eq. (15.100); the result is always $L_0^\perp$ with some constant offset $A = \text{const} \times \mathbb{1}$.

→ Wherever we used $L_0^\perp$ in the *classical* theory, we make the replacement …

$$L_0^\perp \quad \mapsto \quad L_0^\perp - A \tag{15.103}$$

… in the *quantized* theory.

$A = \text{const} \times \mathbb{1}$: Unknown "normal ordering" constant

We henceforth carry the undetermined constant A along; maybe we encounter some condition that constrains A along the way …

The appearance of the undetermined normal ordering constant A might be surprising. However, canonical quantization is not always a unique recipe to bootstrap a quantum theory from a given classical theory. This is only true for the most simple models – if they do not contain terms like $x \cdot p$ that lead to ordering ambiguities. Quantization is not a "fire-and-forget" procedure that assigns every classical theory a unique quantum theory that is magically "true". Classical theories are *limits* (= approximations) of underlying quantum theories for macroscopic systems. As such, they often do not contain enough information to recover the quantum theory unambiguously (↑ *Groenewold's theorem* [318]).

4 | This implies in particular: (remember that $\alpha_0^\mu = \sqrt{2\alpha'}\, p^\mu$ for an open string)

$$2\alpha' p^- = \sqrt{2\alpha'}\alpha_0^- \overset{\substack{15.90\\15.103}}{:=} \frac{1}{p^+}\left(L_0^\perp - A\right) \tag{15.104}$$

Formally, $1/p^+$ is the inverse operator of p^+; it will be canceled → *below* anyway, so that a formal definition is not necessary.

5 | ⊲ Mass shell condition:

With these preparations, we find the quantized version of the mass shell condition Eq. (15.73):

$$M^2 = -p^2 = 2p^+p^- - p_\perp^2 \tag{15.105a}$$

$$\overset{15.104}{=} \frac{1}{\alpha'}\left(L_0^\perp - A\right) - p_\perp^2 \tag{15.105b}$$

$$\overset{15.102}{=} \frac{1}{\alpha'}\left(N^\perp - A\right) \tag{15.105c}$$

This result has two immediate implications:

- The mass of a string depends on the eigenvalues of the level operator $N^\perp$ (= its excitations). This will lead to the identification of various particles in the → *string spectrum* (Section 15.2.2).
- The (so far undetermined) normal ordering constant A *is important*! Its value determines the masses of the particles; in particular – depending on the value of A – the mass squared can become *negative* (which would imply a *space-like* 4-momentum).

6 | ⊲ Virasoro operators:

What is the commutator algebra of the transverse Virasoro operators?

Witt algebra Eq. (15.93) $\xrightarrow{\text{Eq. (15.99)}}$ ⁂ *Virasoro algebra*:

$$\left[L_m^\perp, L_n^\perp\right] \mathrel{\hat{=}} (m-n)L_{m+n}^\perp \underbrace{+\frac{D-2}{12}m(m^2-1)\,\delta_{m+n}}_{\uparrow\text{ Central extension}} \tag{15.106}$$

$D-2 \equiv c$: ↑ *Central charge* (the prefactor $\frac{1}{12}$ is conventional)

- The Virasoro algebra is the most important algebra in string theory. As it descends from the conformal symmetry of the classical action, it is also the centerpiece of more general ↑ *conformal field theories*, where the central charge c is not necessarily linked to the spacetime dimension (this is rather special to string theory).

  It is well-known from conformal field theory that a free scalar (boson) has central charge $c = 1$. Thus, in bosonic string theory, each scalar field X^μ contributes $c = 1$ to the total central charge. In light-cone gauge, there are only $D-2$ transversal fields X^i that are dynamical, so that the total central charge is $c = D-2$.

- For a detailed derivation of Eq. (15.106) see ↑ ZWIEBACH [7] (§12.4, pp. 254–257).

- We found that the Lie algebra of the quantized generators of conformal transformations is different from their classical Poisson algebra Eq. (15.93). [Put differently: The Lie algebra of Virasoro operators does not follow from their classical algebra via the substitution Eq. (15.98).] This *suggests* that the original conformal/Weyl symmetry of the classical action might not be shared by the quantized theory. In general, the phenomenon that a classical symmetry does not survive quantization is called a ↑ *(quantum) anomaly*. In the case of string theory, it is Weyl symmetry that can be spoiled; this particular anomaly is called ↑ *Weyl anomaly*.

- Side note:

  The additional term in Eq. (15.106) is called a ↑ *central extension* of the Witt algebra Eq. (15.93) because it extends the old algebra by a new element of the form const $\times\, \mathbb{1}$ that commutes with all other elements ($L_m^\perp$); such elements (of a group or an algebra) are called ↑ *central* in mathematics. If one exponentiates a centrally extended Lie algebra, the new central element leads to additional phase factors in the multiplication rules of the corresponding Lie group, so called ↑ *cocycles*. These modified multiplication rules define ↑ *projective representations* of the original Lie group (these are essentially group representations "up to phase factors"). Now remember that quantum mechanics is concerned with state vectors in Hilbert spaces *up to global phases*; mathematically speaking, the physical state spaces of quantum theories are ↑ *projective Hilbert spaces*. Physical symmetries on such spaces are then implemented by the aforementioned projective representations. This line of arguments shows that the appearance of central extensions of symmetry algebras in quantum mechanics is directly linked to the fact that global phases are unphysical.