↓Lecture 33 [07.08.24]

5 | ⊲ Hilbert energy-momentum tensor:

$$T_{ab}(\tau,\sigma) \overset{11.106}{=} -\frac{2}{\sqrt{h}}\frac{\delta(\sqrt{h}L_{\mathrm{P}})}{\delta h^{ab}} \overset{11.118}{=} T\left(\partial_a X^\mu \partial_b X_\mu - \tfrac{1}{2}h_{ab}\partial_c X^\mu \partial^c X_\mu\right) \tag{15.32}$$

Use Eq. (11.118) or your result from ➔ Problemset 4 for the Klein-Gordon field to show this. Note that we defined the HEMT here with the opposite sign [cf. Eq. (11.106)]; this is because we use the opposite signature $(-,+)$ for the metric h_{ab}. The sign makes the energy density T_{00} *positive* (which is how the sign of the HEMT is conventionally chosen).

Symmetries constrain the HEMT as follows:

$$\text{Diffeomorphism invariance (15.26)} \quad \xrightarrow{11.109} \quad T^{ab}{}_{;b} \doteq 0 \quad \text{(divergence-free)} \tag{15.33a}$$

$$\text{Weyl invariance (15.28)} \quad \overset{\circ}{\rightarrow} \quad T^a{}_a \equiv 0 \quad \text{(traceless)} \tag{15.33b}$$

To show that the trace vanishes due to Weyl invariance is straightforward. First, define

$$\tilde{h}_{ab} := e^{2\rho}h_{ab}\,, \tag{15.34}$$

and then compute the variational derivative of $\mathcal{L}_{\mathrm{P}}(\tilde{h}_{ab}, X^\mu)$ wrt. ρ:

$$0 \overset{\text{Weyl}}{=} \frac{\delta \mathcal{L}_{\mathrm{P}}}{\delta\rho} \overset{15.28}{=} \frac{\delta \mathcal{L}_{\mathrm{P}}}{\delta\tilde{h}_{ab}}\frac{\delta\tilde{h}_{ab}}{\delta\rho} \overset{11.100}{=} -\sqrt{h}T^{ab}e^{2\rho}h_{ab} = -\sqrt{h}T^a{}_a\,e^{2\rho}\,. \tag{15.35}$$

This implies $T^a{}_a \equiv 0$ without imposing the equations of motion; it is an *identity*. [This is a consequence of the fact that the fields X^μ do not change under Weyl transformations.]

6 | Equations of motion:

- Varying Eq. (15.22) wrt. the world sheet metric h_{ab} yields the HEMT:

$$\delta_h S_{\mathrm{P}} \overset{!}{=} 0 \quad \overset{15.32}{\Longleftrightarrow} \quad T_{ab} \overset{!}{=} 0 \tag{15.36}$$

  T_{ab} has no derivatives of the metric [Eq. (15.32)] → Constraint

- Varying Eq. (15.22) wrt. the fields X^μ yields:

$$\delta_X S_{\mathrm{P}} \overset{!}{=} 0 \quad \Leftrightarrow \quad \partial_a\left(\sqrt{h}h^{ab}\partial_b X^\mu\right) \overset{!}{=} 0 \quad \Leftrightarrow \quad \Delta X^\mu \overset{!}{=} 0 \tag{15.37}$$

  with Laplace-Beltrami operator $\Delta = \nabla^a\nabla_a$ [← Eq. (10.97)].

  Recall ➔ Problemset 4 and Eq. (11.38).

  This EOM looks much more tracktable than the EOM (15.21) of the Nambu-Goto action. But we should not forget that it must be augmented by the constraint Eq. (15.36).

7 | Boundary conditions:

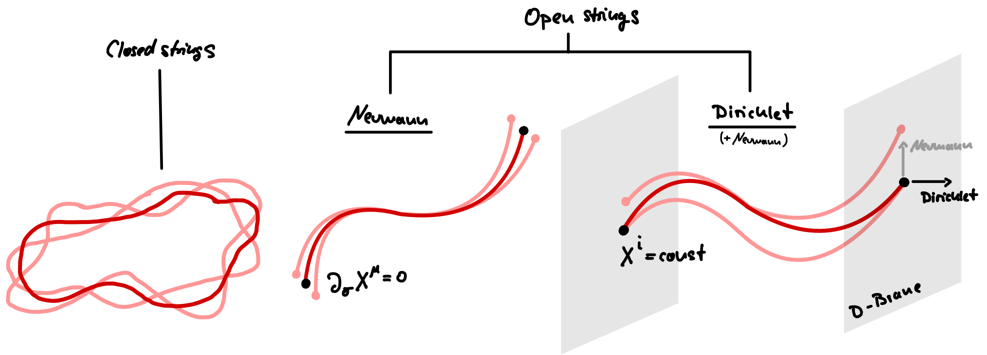
If the world sheet is finite (here in σ direction), the variation of the action has boundary terms that must also vanish (in addition to the EOMs above):

$\triangleleft$ World sheet with $\tau \in \mathbb{R}$ and $\sigma \in I = [0, l]$ for $0 < l < \infty$:

$$\delta_X S_{\mathrm{P}} \stackrel{\circ}{=} \int_{-\infty}^{+\infty} \mathrm{d}\tau \int_0^l \mathrm{d}\sigma \underbrace{\overbrace{\frac{\delta(\sqrt{h} L_{\mathrm{P}})}{\delta X_\mu}}^{\stackrel{!}{=} 0}}_{\to \text{EOM (15.37)}} \delta X_\mu - T \int_{-\infty}^{+\infty} \mathrm{d}\tau \underbrace{\left[\overbrace{\sqrt{h}\,\delta X_\mu\, \partial^\sigma X^\mu}^{\stackrel{!}{=} 0}\right]_{\sigma=0}^{\sigma=l}}_{\text{Boundary term}} \tag{15.38}$$

To show this, use Eqs. (11.98) and (11.100) (for X_μ instead of $g^{\mu\nu}$) and apply them to the Polyakov action Eq. (15.22).

There are three possibilities to make the boundary term vanish:



- Closed string:

  A closed string requires that the points $\sigma = 0$ and $\sigma = l$ in I are identified; in particular:

$$X^\mu(\tau, 0) \stackrel{!}{=} X^\mu(\tau, l) \quad \text{and} \quad \partial^\sigma X^\mu(\tau, 0) \stackrel{!}{=} \partial^\sigma X^\mu(\tau, l) \tag{15.39}$$

  These conditions make the difference of the two boundary terms in Eq. (15.38) vanish. For consistency, also the metric must be periodic: $h_{ab}(\tau, 0) \stackrel{!}{=} h_{ab}(\tau, l)$.

  → All fields are periodic in σ-direction

  → String = Closed loop

- Open string:

  An open string has endpoints; there are two possibilities to make each of the two boundary terms in Eq. (15.38) vanish separately on these endpoints:

  – Neumann boundary conditions:

$$\partial^\sigma X^\mu(\tau, 0) \stackrel{!}{=} 0 \stackrel{!}{=} \partial^\sigma X^\mu(\tau, l) \quad \Leftrightarrow \quad \underbrace{n^a \partial_a X^\mu|_{\partial\mathcal{M}} \stackrel{!}{=} 0}_{\text{Coordinate independent}} \tag{15.40}$$

    Here $\mathcal{M}$ denotes the 2D world sheet and n^a is the normal on the boundary 1D $\partial\mathcal{M}$.

    → Ends of string move *freely* in spacetime

    Note that the *positions* $X^\mu(\tau, 0)$ and $X^\mu(\tau, l)$ are not fixed.

  – Dirichlet boundary conditions:

    The constraints Eq. (15.39) and Eq. (15.40) are Poincaré/Lorentz covariant equations. With these boundary conditions, the internal Poincaré invariance of the theory remains in tact. If we allow for a *violation* of this symmetry, there is a third possibility to make the boundary terms vanish:

$$[\delta X_\mu]_{\sigma=0}^{\sigma=l} \stackrel{!}{=} 0 \quad\Rightarrow\quad \begin{Bmatrix} \partial_\tau X^\mu(\tau,0) = 0 \\ \partial_\tau X^\mu(\tau,l) = 0 \end{Bmatrix} \quad\Rightarrow\quad \begin{Bmatrix} X^\mu(\tau,0) = \text{const} \\ X^\mu(\tau,l) = \text{const} \end{Bmatrix} \tag{15.41}$$

→ String ends are *fixed* (here: in *spacetime*)

For an open string, one can mix Neumann and Dirichlet boundary conditions for the different components X^μ because the boundary terms in Eq. (15.38) are a sum over $\mu = 0, \dots, D-1$. If X^0 and p of the spatial components X^i satisfy *Neumann* boundary conditions, the string can move freely on a p-dimensional hyperplane in space; this hyperplane (extended by one dimension in time) is called a ↑ *Dp-brane*.

→ Strings can be attached to a ↑ *D-branes* (D = Dirichlet)

After quantizing strings attached to a D-brane, one finds that some of their oscillator modes can be interpreted as quantum fluctuations *of the D-brane itself* (their coherent states determine the expecation value of the D-brane position in spacetime). Hence one finds, quite surprisingly, that D-branes are actually *dynamical objects* – and not static & classical background structures.

In the following we only consier closed strings and open strings with Neumann boundary conditions.

8 | Flat gauge: (also called *conformal gauge*)

Mathematical fact: Every two-dimensional pseudo-Riemannian manifold is *conformally flat*:

→ $\forall\, h_{ab}\ \exists$ Coordinates such that

$$h_{ab} = \Omega^2(\tau,\sigma)\eta_{ab} = \Omega^2(\tau,\sigma)\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}_{ab} \tag{15.42}$$

for some non-vanishing *conformal factor* $\Omega(\tau,\sigma)$.

¡! Conformal flatness does not imply the vanishing of the Riemann curvature tensor.

This is a peculiar feature of two dimensions: On a d-dimensional manifold the metric tensor has $d(d+1)/2$ independent components. The diffeomorphism group (coordinate transformations) has d generators [← Eq. (11.101)], which leaves $d(d-1)/2$ degrees of freedom of the metric that cannot be fixed by coordinate transformations. In $d = 2$ this is exactly one degree of freedom, namely the conformal factor in Eq. (15.42).

We can now use the Weyl invariance (15.28) of the Polyakov action to drop the conformal factor:

$$\boxed{\text{Weyl invariance} \quad\rightarrow\quad h_{ab} = \eta_{ab} \qquad ⁂\ \textit{Flat gauge}} \tag{15.43}$$

All calculations that follow are perfomed in flat gauge:

9 | Conjugate momentum & Poisson algebra:

In flat gauge, the Polyakov action & Lagrangian are quite simple:

$$S_{\mathrm{P}}^{\text{flat}}[X] \stackrel{15.22}{=} \frac{T}{2}\int \mathrm{d}\sigma\,\mathrm{d}\tau\, \underbrace{\left[(\dot X)^2 - (X')^2\right]}_{=:\, L_{\mathrm{P}}^{\text{flat}}/(T/2)} \tag{15.44}$$

To prepare for canonical quantization, we need the conjugate momentum of X^μ:

→ Conjugate momentum:

$$\Pi_\mu(\tau,\sigma) := \frac{\partial L_{\mathrm{P}}^{\mathrm{flat}}}{\partial \dot{X}^\mu} = T\dot{X}_\mu \tag{15.45}$$

with satisfies the ↓ *canonical Poisson algebra*: (defined at equal time!)

$$\{X^\mu(\tau,\sigma),\Pi_\nu(\tau,\sigma')\} = \delta(\sigma-\sigma')\,\delta^\mu_\nu \tag{15.46}$$

This is the field-theory analog of $\{x_i, p_j\} = \delta_{ij}$ that you encountered in your course on classical mechanics. The Poisson bracket for fields is defined via functional derivatives. However, we will expand the fields into a discrete set of modes → *below* anyway, so that we can impose this Poisson algebra directly on the modes (without the need for functional derivatives).

10 | Classical solutions of EOM (15.37) for X^μ:

In flat gauge, the Laplace-Beltrami operator yields a simple wave equation:

$$\underbrace{\left(\partial_\tau^2-\partial_\sigma^2\right)}_{\Box} X^\mu = 0 \tag{15.47}$$

We will now write down the general solutions of this EOM for a closed and an open string.

¡! Do not forget that this is only one of the EOMs; it must be augmented by the constraint Eq. (15.36). We will study the implementation of this constraint on the solutions → *later*.

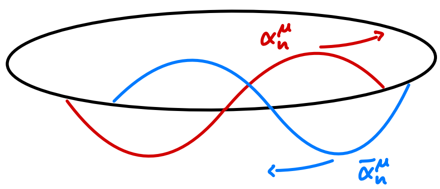
$\overset{\circ}{\rightarrow}$ General solution:

$$X^\mu(\tau,\sigma) = \underbrace{X^\mu_R(\tau-\sigma)}_{\text{“Right mover”}} + \underbrace{X^\mu_L(\tau+\sigma)}_{\text{“Left mover”}} \tag{15.48}$$

Here $X^\mu_{R/L}$ are arbitrary (differentiable) functions of a single variable.

In → *light-cone coordinates* $\sigma^\pm = \tau\pm\sigma$ the EOM (15.47) reads $\partial_+\partial_- X^\mu = 0$. Integrating twice yields the general solution $X^\mu = X^\mu_R(\sigma^-) + X^\mu_L(\sigma^+)$.

i | ◁ Closed string:



a | We must implement the boundary conditions Eq. (15.39) on the solutions Eq. (15.48).

Let *w.l.o.g.* $l=\pi$: (this can always be achieved by reparametrizing the world sheet)

$$X^\mu\in\mathbb{R} \quad\text{and}\quad X^\mu(\tau,\sigma+\pi) = X^\mu(\tau,\sigma)\,. \tag{15.49}$$

→ We want to parametrize *real-valued*, σ*-periodic* and *differentiable* functions.

→ Fourier series

b | $\overset{\circ}{\rightarrow}$ Most general solution:

$$X_R^\mu = \tfrac{1}{2}x^\mu + \alpha' p^\mu(\tau-\sigma) + i\sqrt{\tfrac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}\exp\left[-2in(\tau-\sigma)\right] \tag{15.50a}$$

$$X_L^\mu = \tfrac{1}{2}x^\mu + \alpha' p^\mu(\tau+\sigma) + i\sqrt{\tfrac{\alpha'}{2}}\sum_{n\neq 0}\frac{\tilde\alpha_n^\mu}{n}\exp\left[-2in(\tau+\sigma)\right] \tag{15.50b}$$

Here we use the slope parameter $\alpha' = (2\pi T)^{-1}$ for convenience. Note that we extracted the $n = 0$ component as $\frac{1}{2}x^\mu$ from the sum. The linear part $\propto p^\mu$ is not periodic in σ but becomes so in the sum Eq. (15.48). All prefactors are chosen for convenience.

The solutions are parametrized by the following free parameters:

- $x^\mu, p^\mu \in \mathbb{R}$: Center of mass initial position & momentum of string

  The interpretation of x^μ and p^μ is easily confirmed:

  $$x^\mu \overset{\circ}{=} \frac{1}{\pi}\int_0^\pi d\sigma\, X^\mu(0,\sigma) \tag{15.51}$$

  and

  $$p^\mu \overset{\circ}{=} \frac{1}{2\pi\alpha'}\int_0^\pi d\sigma\, \partial_\tau X^\mu(\tau,\sigma) \overset{15.45}{=} \int_0^\pi d\sigma\, \Pi^\mu(\tau,\sigma)\,. \tag{15.52}$$

- $\alpha_n^\mu, \tilde\alpha_n^\mu \in \mathbb{C}$: Fourier components of string oscillation modes

  Reality condition: $X^\mu \in \mathbb{R} \Leftrightarrow$

  $$\alpha_{-n}^\mu = (\alpha_n^\mu)^* \quad\text{and}\quad \tilde\alpha_{-n}^\mu = (\tilde\alpha_n^\mu)^* \tag{15.53}$$

  When checking this, do not forget that $n \in \mathbb{Z}$ so that $n \mapsto -n$.

It will be convenient to define for the closed string: $\alpha_0^\mu \equiv \tilde\alpha_0^\mu \equiv \sqrt{\frac{\alpha'}{2}}\,p^\mu$.

c | Poisson algebra Eq. (15.46) in mode space $\overset{\circ}{\Leftrightarrow}$

$$\{x^\mu, p^\nu\} = \eta^{\mu\nu} \tag{15.54a}$$

$$\{\alpha_m^\mu, \alpha_n^\nu\} = im\delta_{m+n}\eta^{\mu\nu} \tag{15.54b}$$

$$\{\tilde\alpha_m^\mu, \tilde\alpha_n^\nu\} = im\delta_{m+n}\eta^{\mu\nu} \tag{15.54c}$$

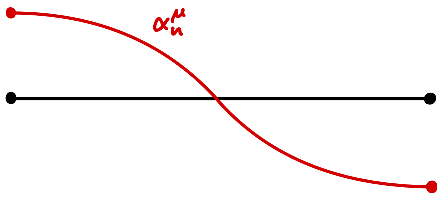
$$\{\alpha_m^\mu, \tilde\alpha_n^\nu\} = 0 \tag{15.54d}$$

- ¡! Note the complex i in the Poisson algebra of the Fourier modes. After quantization, this will make pairs $(\alpha_n^\mu, \alpha_{-n}^\mu)$ into creation and annihilation operators of a harmonic oscillator mode.
- The Poisson algebra is the starting point for a canonical quantization procedure (→ *below*). So what is the point of the *classical* solutions Eq. (15.50)? There are two aspects to consider:

First, for $\tau = 0$, the expansion Eq. (15.50) is a completely general parametrization of configurations X^μ of the string that are consistent with its boundary conditions. This makes the Fourier coefficients $\alpha_n^\mu, \tilde{\alpha}_n^\mu$ [together with x^μ and the reality constraint Eq. (15.53)] a convenient (and discrete) set of dynamical variables to encode the field X^μ. The Fourier expansion exploits the symmetry of the problem under translations along σ, and leads to a decoupling of the Poisson brackets between different modes. (Note that only brackets of the form $\{\alpha_n^\mu, \alpha_{-n}^\mu\}$ do *not* vanish.)

Second, eventually we want to quantize the fields X^μ. Since the Heisenberg field operators of free fields obey the *classical* equations of motion (↑ *Quantum field theory* [20]), we can simply quantize the mode operators and plug them into Eq. (15.50) to obtain the Heisenberg field operators for $\tau \neq 0$ (thereby skipping the solution of the Heisenberg equation, i.e., the application of the time-evolution operator).

ii | ⊲ Open string & Neumann boundary conditions: (no D-branes!)



a | We must implement the boundary conditions Eq. (15.40) on the solutions Eq. (15.48).

Let again *w.l.o.g.* $l = \pi$:

$$X^\mu \in \mathbb{R} \quad \text{and} \quad \partial_\sigma X^\mu(\tau, 0) = 0 = \partial_\sigma X^\mu(\tau, \pi) \tag{15.55}$$

b | $\overset{\circ}{\rightarrow}$ General solutions:

$$X^\mu(\tau, \sigma) = x^\mu + 2\alpha' p^\mu \tau + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{\alpha_n^\mu}{n} \exp[-in\tau] \cos(n\sigma) \tag{15.56}$$

You can derive this from the closed string solutions Eq. (15.50) by imposing the constraint Eq. (15.55) which cuts the degrees of freedom in half.

The solutions are parametrized by the following free parameters:

- $x^\mu, p^\mu \in \mathbb{R}$: Center of mass initial position & momentum of string
- $\alpha_n^\mu \in \mathbb{C}$: Fourier components of string oscillation modes

  → Only one set $\{\alpha_n^\mu\}$ of oscillator modes!

  Reality condition: $X^\mu \in \mathbb{R}$ ⇔

$$\boxed{\alpha_{-n}^\mu = (\alpha_n^\mu)^*} \tag{15.57}$$

It will be convenient to define for the closed string: $\alpha_0^\mu \equiv \sqrt{2\alpha'} p^\mu$

(Note that this definition is different from the closed string!)

c | Poisson algebra Eq. (15.46) in mode space $\overset{\circ}{\Leftrightarrow}$

$$\{x^\mu, p^\nu\} = \eta^{\mu\nu} \quad \text{and} \quad \{\alpha_m^\mu, \alpha_n^\nu\} = im\delta_{m+n}\eta^{\mu\nu} \tag{15.58}$$

This is the subset of Eq. (15.54) where the modes $\tilde{\alpha}_n^\mu$ have been dropped.

11 | ⊲ Constraint Eq. (15.36):

Now that we have the solutions of the EOM (15.37) (for open and closed strings), we should also impose the constraint Eq. (15.36) on them. Here we only simplify the contraint in flat gauge, but do not enforce it yet on the level of oscillator modes. We do this → *later* after some more gauge fixing.

$$T_{ab}(\tau,\sigma) \overset{15.32}{\underset{15.43}{=}} T\left(\partial_a X^\mu \partial_b X_\mu - \tfrac{1}{2}\eta_{ab}\eta^{cd}\partial_c X^\mu \partial_d X_\mu\right) \overset{!}{=} 0 \tag{15.59}$$

$\overset{\circ}{\rightarrow}$ In components this reads:

$$T_{00} = T_{11} = \tfrac{1}{2}\left[(\dot{X})^2 + (X')^2\right] \overset{!}{=} 0 \tag{15.60a}$$

$$T_{01} = T_{10} = \dot{X} \cdot X' \overset{!}{=} 0 \tag{15.60b}$$

We can now check Eq. (15.33b) explicitly:

$$T^a{}_a = \eta^{ab}T_{ab} = T_{11} - T_{00} = 0 \tag{15.61}$$

As explained above, this is a consequence of the Weyl invariance of the Polyakov action.

The constraint equations can be combined in a convenient form:

$$\text{Eq. (15.60)} \quad \Leftrightarrow \quad \left(\dot{X} \pm X'\right)^2 \overset{!}{=} 0 \tag{15.62}$$

This will be our starting point to enforce the constraint → *later*.

12 | Conserved quantities:

As preparation for → *later*, let us briefly discuss the conserved quantities that follow from the global Poincaré symmetry of the Polyakov action:

Poincaré symmetry Eq. (15.24) ⟶ Noether currents = ⁂ *World sheet currents*

[Remember: Poincaré transformations = Translations + Rotations + Boosts]

¡! The Poincaré symmetry is an *internal* symmetry, and the corresponding Noether currents live on the 2D *world sheet* (not on spacetime!). This means that latin indices $a, b, \ldots$ label the *components* of the currents, whereas greek (spacetime) indices $\mu, \nu, \ldots$ label different *types* of currents, corresponding to different spacetime symmetries.

- μ-Translations: $\delta_\nu X^\mu = \delta_\nu^\mu$

  (Recall Eqs. (6.79) and (6.89) and note that a μ-translation shifts the *value* of the field X^μ.)

$$\text{Eq. (6.84)} \overset{\circ}{\rightarrow} \quad P_a^\mu = T\partial_a X^\mu \quad \text{with} \quad \partial_a P^{a\mu} \doteq 0 \tag{15.63}$$

⟶ Conserved charge: Total 4-momentum:

$$P^\mu \overset{6.86}{:=} \int_0^\pi d\sigma P_0^\mu = T\int_0^\pi d\sigma \dot{X}^\mu \overset{15.52}{=} p^\mu \tag{15.64}$$

When using Eq. (6.84) to derive this, be very careful: Here the symmetry is labeled by a spacetime index ($a \mapsto \nu$) whereas "spacetime" is now the world sheet ($\mu \mapsto a$). The field is still a scalar, but there are D of them labeled by another spacetime index ($\phi \mapsto X^\mu$). Since the Poincaré symmetry is an *internal* symmetry, it is $\delta_a x^\mu \mapsto \delta_\nu \sigma^a = 0$, i.e., it does not transform the world sheet coordinates.

- $\mu\nu$-Rotations: $\delta^{\alpha\beta} X^\mu = \eta^{\alpha\mu} X^\beta - \eta^{\beta\mu} X^\alpha$ [← Eq. (6.78), we drop the arbitrary $\frac{1}{2}$]
  ("Rotations" here refers to both spatial rotations and boosts.)

$$\text{Eq. (6.84)} \overset{\circ}{\to} \qquad J_a^{\mu\nu} = T\left(X^\mu \partial_a X^\nu - X^\nu \partial_a X^\mu\right) \quad \text{with} \quad \partial_a J^{a\mu\nu} \doteq 0 \tag{15.65}$$

→ Conserved charge: Total 4-angular momentum:

$$J^{\mu\nu} = \int_0^\pi \mathrm{d}\sigma J_0^{\mu\nu} = T \int_0^\pi \mathrm{d}\sigma \left(X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu\right) \tag{15.66}$$

After quantization, this charge becomes an *operator* that generates rotations & boosts on the Hilbert space of the string (just like the momentum operator generates translations). It will be crucial to determine the critical dimension of bosonic string theory.

13 | Hamiltonian:

In flat gauge, and with the mode expansion at hand, it is now straightforward to derive the Hamiltonian of the Polyakov action:

i | As usual, we get the Hamiltonian via Legendre transformation from the Polyakov Lagrangian:

$$H = \int_0^\pi \mathrm{d}\sigma \left[\dot{X}\cdot\Pi - L_{\mathrm{P}}^{\mathrm{flat}}\right] \overset{15.44}{\underset{15.45}{=}} \frac{T}{2}\int_0^\pi \mathrm{d}\sigma \left[(\dot{X})^2 + (X')^2\right] \tag{15.67}$$

Using the Fourier expansion of the fields, this can be rewritten in terms of oscillator modes:

$$\text{Open string:} \quad H \overset{15.56}{=} \frac{1}{2}\sum_n \alpha_{-n}\cdot\alpha_n \tag{15.68a}$$

$$\text{Closed string:} \quad H \overset{15.48}{\underset{15.50}{=}} \frac{1}{2}\sum_n \left(\alpha_{-n}\cdot\alpha_n + \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n\right) \tag{15.68b}$$

- Here we introduced the shorthand notation $\alpha_n \cdot \alpha_{-n} \equiv \eta_{\mu\nu}\,\alpha_n^\mu \alpha_{-n}^\nu$.
- Note that these sums include the $n = 0$ mode, i.e., the momentum p^μ of the string:

$$\text{Open string } (\alpha_0^\mu = \sqrt{2\alpha'}\,p^\mu): \qquad \tfrac{1}{2}\alpha_0\cdot\alpha_0 = \alpha' p^2 \tag{15.69a}$$

$$\text{Closed string } (\tilde{\alpha}_0^\mu = \sqrt{\alpha'/2}\,p^\mu): \quad \tfrac{1}{2}\left(\alpha_0\cdot\alpha_0 + \tilde{\alpha}_0\cdot\tilde{\alpha}_0\right) = \tfrac{1}{2}\alpha' p^2 \tag{15.69b}$$

  These terms account for the kinetic energy of the string.
- To derive Eq. (15.68), use that for $n, m \in \mathbb{N}$

$$\int_0^\pi \mathrm{d}\sigma \cos(n\sigma)\cos(m\sigma) = \frac{\pi}{2}\delta_{n,m} = \int_0^\pi \mathrm{d}\sigma \sin(n\sigma)\sin(m\sigma)\ . \tag{15.70}$$

ii | The constraint equation implies that the Hamiltonian vanishes on-shell:

$$\text{Eqs. (15.60a) and (15.67)} \quad \Rightarrow \quad H \doteq 0 \tag{15.71}$$

This is similar to Section 5.4 [in particular Eq. (5.93)] were we found the Hamiltonian of the relativistic particle to vanish as well. We identified the reparametrization invariance as the root cause, which is a local (gauge) symmetry that produces constraints via Noether's second theorem. Here, the Hamiltonian generates translations in τ – but τ is only one of many possible time-like parametrizations (due to the diffeomorphism invariance on the world sheet); it has no physical interpretation. Consequently, the Hamiltonian that generates translations in this parameter has no physical significance either.

iii | Eq. (15.71) → Mass shell condition:

We study open and closed strings separately:

- ◁ *Open* string: Combining our previous results implies:

$$\tfrac{1}{2}\alpha_0 \cdot \alpha_0 + \sum_{n>0} \alpha_{-n} \cdot \alpha_n \overset{15.68a}{\underset{15.71}{=}} 0 \quad \Leftrightarrow \quad \alpha' p^2 \overset{15.69a}{=} -\sum_{n>0} \alpha_{-n} \cdot \alpha_n \tag{15.72}$$

  Thus the norm of the 4-momentum of the string is determined by its oscillation modes.

  → Recall that the norm of a 4-momentum is a Lorentz scalar called *(rest) mass*:

$$p^2 \overset{5.4}{=} -M^2 \quad \Rightarrow \quad M^2 = \frac{1}{\alpha'} \sum_{n>0} \alpha_{-n} \cdot \alpha_n \tag{15.73}$$

  M: Rest mass of the *open* string

  - If you think about it, this result makes sense: The oscillations of the string contribute to its *internal* energy. And in Section 5.2 we argued that in a relativistic theory, any type of internal energy contributes to the rest mass of an object.
  - Note that $(\alpha_n^\mu)^* = \alpha_{-n}^\mu$ makes terms like $\alpha_{-n}^\mu \alpha_n^\mu = |\alpha_n^\mu|^2$ non-negative. However, not also that $\alpha_{-n} \cdot \alpha_n = \eta_{\mu\nu}\, \alpha_{-n}^\mu \alpha_n^\nu$, so that the Lorentzian signature of $\eta_{\mu\nu}$ produces positive and negative terms in the sum. The current form of Eq. (15.73) is therefore potentially problematic, since the left-hand side is the mass squared.

- ◁ *Closed* string: Along the same lines, one finds for the closed string the constraint:

$$\tfrac{1}{2}\alpha' p^2 \overset{15.68b,\ 15.71,\ 15.69b}{=} -\sum_{n>0} (\alpha_{-n} \cdot \alpha_n + \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n) \tag{15.74}$$

  … so that the rest mass of the string is given by:

$$p^2 \overset{5.4}{=} -M^2 \quad \Rightarrow \quad M^2 = \frac{2}{\alpha'} \sum_{n>0} (\alpha_{-n} \cdot \alpha_n + \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n) \tag{15.75}$$

  M: Rest mass of the *closed* string