↓Lecture 32 [06.08.24]

15. Sneak Peek: Bosonic String Theory

This primer on bosonic string theory is an amalgamation of various sources, mostly lecture scripts (by Carmen A. Núñez, Arthur Hebecker, and David Tong), and the introductory textbook by BARTON ZWIEBACH [7].

1 | *What is the rationale of string theory?*

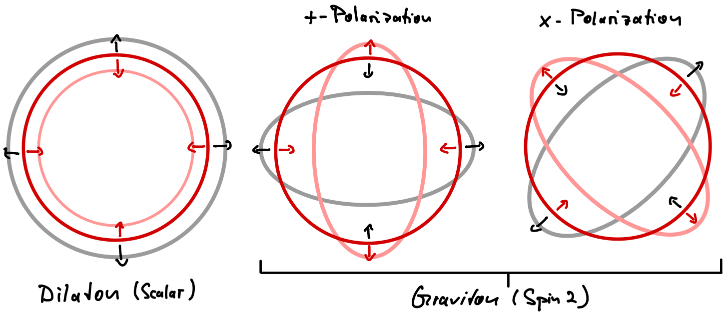
- Hypotheses:
  - $\sphericalangle$ Fixed background spacetime $g_{\mu\nu}$

    In contrast to GENERAL RELATIVITY, string theory has *no* manifestly background independent formulation. The dynamics of the spacetime geometry is described by quantum fluctuations (of gravitons) on top of a classical, static background metric.
  - Postulate elementary entities: Relativistic strings

    The strings of string theory are *elementary entities* that propagate (and interact) on the fixed background spacetime; think of them as "rubber bands," i.e., they can be stretched. These strings can be closed (= loops) or open (→ *later*). Note that strings are not emergent from other degrees of freedom – string theory does not explain were strings come from.
  - Postulate an action that determines ...
    * ... dynamics of single string and
    * ... interaction between strings.

    This action is motivated as a generalization of the action of a free point particle.
  - Hope:
    * Quantized excitations of strings = Fundamental particles
    * Joining/splitting of strings = Fundamental interactions between particles
    * No point-like particles $\rightarrow$ No UV-divergences for $\Lambda \rightarrow \infty$
    * Classical limit $\rightarrow$ GENERAL RELATIVITY
- Intuition:

  Closed strings can oscillate. Their lowest-frequency modes look as follows:



Note that the left oscillation is invariant under rotations about the symmetry axis, whereas the two modes on the right transform into each other under rotations by $\pm 45°$ – just as a gravitational wave with helicity ± 2 would (← Section 13.4).

→ We should expect a graviton state from *closed* strings!

- At this point, it is unclear whether these modes are truly *massless* after quantization (as required for excitations of a long-range interaction like gravity).
- The "breathing mode" corresponds to a scalar particle called → *dilaton* which comes along with the graviton in string theory; this means that string theory actually predicts a *scalar-tensor theory* of gravity (← Section 12.3). For consistence with reality (in GENERAL RELATIVITY there is no dilaton), there must be a mechanism to render the dilaton massive (= short-ranged).

2 | *How to identify gravity?*

Since string theory follows Approach 1, we will not start from GENERAL RELATIVITY and the Einstein-Hilbert action. But how do we know then that string theory is actually a quantum theory of *gravity*? How do we identify the "gravity" part? We could of course hope that the Einstein field equations fall into our lap, but this is naïve. String theory is a *quantum* theory and the EFEs are *classical* – and the classical limit of a quantum theory is often not evident at all.

A quantum theory of gravity should somehow quantize the gravitational field of GENERAL RELATIVITY, i.e., the metric tensor field $g_{\mu\nu}$ that describes the geometry of spacetime. To identify gravity is then tantamount to identifying the (excitations/quanta of the) metric. Hence we arrive at the fundamental question what makes a field "the metric" in the first place? Up to now we always postulated the existence of a Riemannian manifold equipped with a metric tensor. We show below that this is not necessary. A field does not become "the metric" by *declaration*, but by the way it *interacts* with other fields. This yields an operational method to identify gravity in any theory:

i | Observation (Section 13.4): Gravitational waves …

- … propagate with the speed of light.
- … have helicity ± 2.

→ Gravitons should be *massless spin-2* particles.

→ If we want to "find gravity" we should search for *massless rank-2* tensor fields $h_{\mu\nu}$.

Since $1 \otimes 1 = 0 \oplus 1 \oplus 2$ (irreducible representations of SO(3), ↓ *angular momentum coupling*), spin-2 particles are described by rank-2 tensor fields: $h_{\mu\nu}$. This makes sense if you recall Eq. (13.183) which directly links the two spacetime-indices of the tensor field with the helicity ± 2 under spatial rotations. It also makes sense because a metric tensor $g_{\mu\nu}$ is a (symmetric) rank-2 tensor field. That the field is *massless* is also directly related to the fact that gravity is a *long-range* interaction (like electromagnetism).

→ *Question:* What makes a massless rank-2 tensor field "the metric"?

ii | $\sphericalangle$ Massless rank-2 tensor field $h_{\mu\nu}$ on static Minkowski space $\eta_{\mu\nu}$:

The field $h_{\mu\nu}$ is not (yet) the metric – it is just an ordinary tensor field on Minkowski space!

Massless → Only quadratic derivatives allowed $\overset{\circ}{\to}$ Four possible terms: (Details:↑ Ref. [313])

$$\partial^\nu h_{\mu\nu}\partial_\sigma h^{\mu\sigma}\,,\quad \partial^\nu h_{\mu\nu}\partial^\mu h_\sigma{}^\sigma\,,\quad \partial^\mu h_\nu{}^\nu \partial_\mu h_\sigma{}^\sigma\,,\quad \partial_\sigma h_{\mu\nu}\partial^\sigma h^{\mu\nu} \tag{15.1}$$

- All other conceivable contractions are related to these terms modulo partial integration (= total derivatives).
- Terms without derivative like $h_{\mu\nu}h^{\mu\nu}$ lead to a constant shift in the relativistic dispersion of the field theory, i.e., to *massive* excitations.

iii | $\sphericalangle$ Additional matter fields ϕ: $\mathcal{L}_{\text{Matter}}(\phi,\partial\phi) \to$ HEMT $T^{\mu\nu}_{\text{Matter}}$ with $\partial_\nu T^{\mu\nu}_{\text{Matter}} \doteq 0$

Assumption:

$h_{\mu\nu}$ couples to energy & momentum via $T^{\mu\nu}_{\text{matter}}$.

This is our *only* assumption that makes the massless rank-2 tensor field "special". We will see that this assumption (plus some self-consistency condition) is all that is needed to elevate $h_{\mu\nu}$ from an ordinary tensor field to "the metric".

→ Most general action:

$$S[h,\phi] := \int \mathrm{d}^4x \Bigg[\begin{Bmatrix} a\,\partial^\nu h_{\mu\nu}\partial_\sigma h^{\mu\sigma} + b\,\partial^\nu h_{\mu\nu}\partial^\mu h_\sigma{}^\sigma \\ +c\,\partial^\mu h_\nu{}^\nu \partial_\mu h_\sigma{}^\sigma + d\,\partial_\sigma h_{\mu\nu}\partial^\sigma h^{\mu\nu} \end{Bmatrix} + \tfrac{\kappa}{2} h_{\mu\nu}T^{\mu\nu}_{\text{Matter}} + \mathcal{L}_{\text{Matter}} \Bigg] \tag{15.2}$$

with arbitrary couplings $a,b,c,d,\kappa \in \mathbb{R}$.

Since $T^{\mu\nu}_{\text{Matter}}$ is symmetric, the tensor field is *w.l.o.g.* symmetric as well: $h_{\mu\nu} = h_{\nu\mu}$.

iv | $\overset{\circ}{\to}$ Equation of motion for $h_{\mu\nu}$:

$$\underbrace{[\dots\partial^2 h\dots]^{\mu\nu}}_{\text{Depends on } a,b,c,d} = \tfrac{\kappa}{2} T^{\mu\nu}_{\text{Matter}} \tag{15.3}$$

This EOM is linear in $h_{\mu\nu}$ since the action is quadratic.

→ Energy-momentum conservation:

$$\partial_\nu[\dots\partial^2 h\dots]^{\mu\nu} = \tfrac{\kappa}{2}\partial_\nu T^{\mu\nu}_{\text{Matter}} \doteq 0 \tag{15.4}$$

We want this to be *identically* satisfied for $h_{\mu\nu}$:

$$\partial_\nu[\dots\partial^2 h\dots]^{\mu\nu} \equiv 0 \quad \overset{\circ}{\Rightarrow} \quad a = \tfrac{1}{2},\ b = -\tfrac{1}{2},\ c = \tfrac{1}{4},\ d = -\tfrac{1}{4} \tag{15.5}$$

The solution is unique up to a global rescaling that can be absorbed into κ via a rescaling of the tensor field.

This means that energy momentum conservation for the matter fields is *enforced* by the coupling to $h_{\mu\nu}$ rather than a constraint on the dynamics of $h_{\mu\nu}$ itself.

v | Hence we end up with the most general action that meets our requirements:

$$S[h,\phi] = \int \mathrm{d}^4x \Big[\underbrace{\mathcal{L}_0(h,\partial h)}_{\leftarrow \text{Eq. (14.19)}} + \tfrac{\kappa}{2} h_{\mu\nu} T^{\mu\nu}_{\text{Matter}} + \mathcal{L}_{\text{Matter}} \Big] \tag{15.6}$$

¡! The quadratic Lagrangian of the tensor field is identical to $\mathcal{L}_0$ in Eq. (14.19) (in which $h_{\mu\nu}$ is the deviation of the metric $g_{\mu\nu}$ from Minkowski space $\eta_{\mu\nu}$). Recall that Eq. (14.19) was derived from the Einstein-Hilbert action Eq. (14.16). This result shows that the seemingly arbitrary structure of $\mathcal{L}_0$ in Eq. (14.19) is actually not arbitrary at all – it is the *only* possible quadratic action for a massless rank-2 tensor field that couples to a conserved current.

vi | Inconsistency of Eq. (15.6):

Eq. (15.6) is conceptually inconsistent because it implies the existence of two "types" of energy & momentum: The first type is the energy & momentum of matter fields, which couples to $h_{\mu\nu}$. But $h_{\mu\nu}$ is a dynamical field and therefore carries energy & momentum of its own – to this second type $h_{\mu\nu}$ does *not* couple. This doesn't make sense and we should get rid of this two-class society of energy & momentum:

This is not just a conceptual inconsistency: Enforcing energy-momentum conservation on a subsystem (the matter fields) while coupling this subsystem to another dynamical field (the tensor field) cannot be consistent (i.e., allow for solutions of the combined EOMs). You studied this on ➔ Problemset 1.

→ *Assumption (updated):*

$h_{\mu\nu}$ couples to energy & momentum *of all fields* (including itself!).

→ We therefore should replace the matter HEMT by the total HEMT of the theory:

$$T^{\mu\nu}_{\text{Matter}} \mapsto T^{\mu\nu} := T^{\mu\nu}_{\text{Matter}} + T^{\mu\nu}_{h} \tag{15.7}$$

But this makes the Lagrangian self-referential: $T^{\mu\nu}_h$ is computed from the part of the Lagrangian that includes $h_{\mu\nu}$ – which includes $T^{\mu\nu}_h$. Thus you will be forced to add higher and higher order terms of $h_{\mu\nu}$ to make $h_{\mu\nu}$ couple to its own energy-momentum tensor. This infinite series can be summed and yields a new, non-linear theory for the tensor field $h_{\mu\nu}$: the Einstein-Hilbert action!

One can show [102] $\overset{*}{\rightarrow}$

$S[h,\phi]$ becomes the Einstein-Hilbert action of the field $g_{\mu\nu} := \eta_{\mu\nu} + h_{\mu\nu}$ which couples minimally to $\mathcal{L}_{\text{Matter}}$:

$$\text{Eq. (15.6)} \quad \mapsto \quad \text{Eq. (12.65)}$$

This means that $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ *becomes* the metric – while the static background $\eta_{\mu\nu}$ becomes unobservable – simply by demanding that the massless tensor field $h_{\mu\nu}$ couples to the energy & momentum of all fields.

vii | Conclusion:

$$\boxed{\left.\begin{matrix}\text{Massless tensor } h_{\mu\nu} \\ \ldots \text{ that couples to } T^{\mu\nu}\end{matrix}\right\} \Rightarrow \underbrace{\left\{\begin{matrix} g_{\mu\nu} \equiv \eta_{\mu\nu} + h_{\mu\nu} : \text{Metric} \\ \text{\& Einstein-Hilbert action}\end{matrix}\right.}_{\text{GENERAL RELATIVITY}}}$$

- So in principle, we should look for quantized, massless excitations that transform under a spin-2 representation (have two symmetric spacetime-indices). If these excitations couple to energy & momentum, they are the excitations of the metric field, i.e., gravitons.

  We will not do the latter in string theory, as it requires studying the interactions of strings. However, we will use a different argument to show that the gravitons of string theory have metric meaning.
- Let me reformulate the conclusion of this part, as its importance cannot be overstated:

  Imagine you are given the action of everything Eq. (12.65), but all fields (metric and matter alike) have been labeled X^i were i runs through all components of all fields; for good measure, all interactions are given by one big sum of many terms. To interpret this theory, do you need to know which field plays the role of the metric of spacetime? According to our findings above, the surprising answer is "No": The metric field $g_{\mu\nu}$ is not the metric *by declaration* – it behaves as the metric because it can be interpreted as a massless rank-2 tensor that couples to the total energy-momentum tensor. Being the metric means that the values of the field correlate with the relational properties we call "length" and "time", and these correlations are established by its coupling to the energy-momentum tensor (which, unsurprisingly, generates local translations in space and time).
- The above line of arguments shows again that the Einstein field equations are very *generic*: One does not need much input to end up with GENERAL RELATIVITY; recall Section 12.1.

3 | Reminder (← Section 5.3): Relativistic point particle:

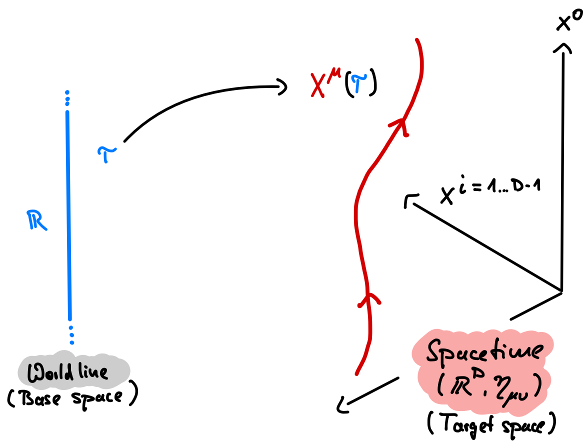
¡! Throughout this chapter we consider objects on D dimensional *Minkowski space*.

i | $\sphericalangle$ Action on D-dimensional Minkowski space $\mathbb{R}^{1,D-1}$:

$$S[X] = -m \int \mathrm{d}\tau \sqrt{-\dot{X}^\mu \dot{X}_\mu} \quad \propto \quad \text{Proper time along trajectory } X^\mu \tag{15.8}$$

This is a functional of time-like trajectories $X^\mu : \mathbb{R} \to \mathbb{R}^{1,D-1}$:

In string theory, points in spacetime are conventionally denoted by capital letters: X^μ.



This action Eq. (15.8) is …

- Poincaré/Lorentz invariant
- Reparametrization invariant $[\tau \mapsto \tilde{\tau} = \tilde{\tau}(\tau)]$ (← Section 5.4)

ii | → Canonical momentum:

$$p_\mu := \frac{\partial L}{\partial \dot{X}^\mu} = \frac{m\dot{X}_\mu}{\sqrt{-\dot{X}^\mu \dot{X}_\mu}} \tag{15.9}$$

Note: These D momenta are not independent! → Constraint:

$$p^2 = -m^2 \qquad \text{(Mass shell condition)} \tag{15.10}$$

iii | The action Eq. (15.8) is not easy to work with because of the square root. Can one get rid of it?

→ ⊲ Alternative action with *auxiliary variable* $e = e(\tau)$:

$$S[e, X] := \frac{1}{2}\int \mathrm{d}\tau \left[e^{-1}\dot{X}^\mu \dot{X}_\mu - em^2\right] \tag{15.11}$$

→ Classically equivalent to Eq. (15.8)

To check this, compute the EOM for the auxiliary variable e,

$$\frac{\partial L}{\partial e} = 0 \quad \Leftrightarrow \quad e^2 = -\frac{1}{m^2}\dot{X}^\mu \dot{X}_\mu\,, \tag{15.12}$$

and plug this back into Eq. (15.11) which immediately yields Eq. (15.8).

Benefits of Eq. (15.11):

- No square root.
- Well-defined for massless particles (= null trajectories).
- Quadratic in derivatives → Quantization via path integral straightforward.

For these reasons, we will use a similar construction for the relativistic string → *below*.

15.1. The classical relativistic string

4 | Relativistic string:

We generalize the relativistic point particle (which traces out a 1D world line in spacetime) to a relativistic string (which traces out a 2D world *sheet*):

¡! This is still classical, relativistic physics; there is no quantum mechanics involved!

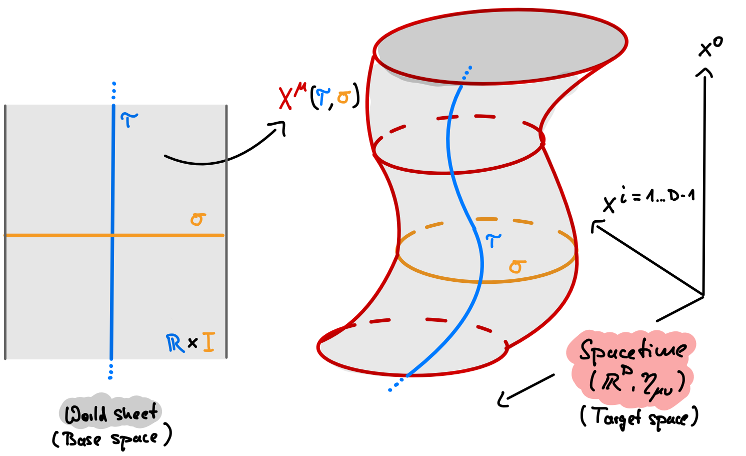
i | ⊲ Parametrization of *2D world sheet* in D spacetime dimensions:

$$X^\mu : \underbrace{\mathbb{R}\times I}_{\text{World sheet}} \to \underbrace{\mathbb{R}^{1,D-1}}_{\text{Spacetime}} \quad \text{with} \quad (\tau,\sigma) \mapsto X^\mu(\tau,\sigma) \tag{15.13}$$

$I \subseteq \mathbb{R}$: Some interval for σ (will be specified later)

Interpretation of world sheet coordinates:

- τ: Time (coordinate) along trajectory of string
- σ: Point (coordinate) on string



¡! You should think of the parameter range as the *base space* and spacetime as the *target space*; the string positions X^μ are then D *fields* on the 2D base space (the world sheet). This implies that the first-quantized theory of a relativistic string will be a 1+1-dimensional *quantum field theory*.

ii | What is a reasonable String action?

We do not derive this action but motivate it as generalization of the relativistic point particle:

$\sphericalangle$ Tangent vectors to world sheet embedded in spacetime:

$$X_\tau := \underbrace{\partial_\tau X^\mu}_{=:\dot{X}^\mu}\partial_\mu \quad \text{and} \quad X_\sigma := \underbrace{\partial_\sigma X^\mu}_{=:X'^\mu}\partial_\mu \tag{15.14}$$

→ Induced ⁂ *world sheet metric*:

$$g_{ab} := \eta(X_a, X_b) = \partial_a X^\mu \partial_b X_\mu \tag{15.15}$$

with $a, b \in \{\tau, \sigma\} \equiv \{0, 1\}$.

The world sheet is a 2D submanifold of D-dimensional spacetime. The Minkowski metric $\eta_{\mu\nu}$ then induces a metric on the world sheet; just like the surface of a ball inherits a metric from the Euclidean space in which it is embedded.

→ World sheet area element:

$$\mathrm{d}A = \sqrt{|\det g_{ab}|}\,\mathrm{d}\sigma\,\mathrm{d}\tau \tag{15.16}$$

- Integrating this over τ and σ yields the surface area of the world sheet wrt. the Minkowski metric of spacetime.
- This is a mathematical fact from Riemannian geometry; it has nothing to do with string theory. Remember that the determinant of a 2×2-matrix is the area of a parallelogram determined by the four numbers of the matrix. You can also think of the worldsheet as a two-dimensional Riemannian manifold. From Eq. (10.101) it follows then that the coordinate-independent volume form on such a manifold is $\mathrm{d}V = \sqrt{g}\mathrm{d}^d x = \sqrt{\bar{g}}\mathrm{d}^d\bar{x}$; in $d = 2$ dimensions this is simply the area: $\mathrm{d}A = \sqrt{g}\mathrm{d}^2 x$.

  For details: ↑ ZWIEBACH [7] (§6.1-§6.3, pp. 100–110).

Remember: Relativistic particle action (15.8) $\propto$ Length of world line

→ Idea: Relativistic string action $\propto$ *Area* of *world sheet*

→ ⁂ *Nambu-Goto action*:

$$\boxed{S_{\mathrm{NG}}[X] := -T\int \mathrm{d}A \overset{15.15}{\underset{15.16}{=}} -T\int\sqrt{(\dot{X}\cdot X')^2-(\dot{X})^2(X')^2}\,\mathrm{d}\sigma\mathrm{d}\tau} \tag{15.17}$$

Note that $\dot{X}$ is time-like whereas X' is space-like, so that the expression under the square root is positive. Expressions like $\dot{X}\cdot X'$ are short for $\dot{X}^\mu X'_\mu$ etc.

¡! This is a classical 2D field theory on the world sheet with D fields: $X^\mu(\tau,\sigma)$.

There is only one parameter that will show up in various permutations:

- T: ⁂ *String tension*
- $\alpha' \equiv \frac{1}{2\pi T}$: ⁂ *Regge slope*
- $\ell \equiv \sqrt{2\alpha'} = \frac{1}{\sqrt{\pi T}}$: ⁂ *String length*

iii | Symmetries of S_{NG}:

It is always good to know the symmetries of a theory, both global and local (gauge):

- D-dimensional Poincaré invariance:

$$\tilde{X}^\mu(\tau,\sigma) = \Lambda^\mu{}_\nu X^\nu(\tau,\sigma) + a^\mu \tag{15.18}$$

  - This follows from Eq. (15.17) because the integrand is a scalar and Poincaré transformations are isometries of Minkowski space. This symmetry is therefore a consequence of our chosen background spacetime.
  - This is a *global* symmetry on the world sheet as it transforms the fields independent of the point (τ,σ) on the world sheet. It is also an *internal* symmetry, in that it mixes only the components of the fields and does not mess with the world sheet points. (Recall that the world sheet – and not spacetime – is the base space of our field theory!)

- Reparametrization invariance = 2D diffeomorphism invariance:

$$\tilde{X}^\mu(\tilde{\tau},\tilde{\sigma}) := X^\mu(\tau,\sigma) \quad \text{with} \quad \tilde{\tau} = \tilde{\tau}(\tau,\sigma) \text{ and } \tilde{\sigma} = \tilde{\sigma}(\tau,\sigma) \tag{15.19}$$

  - This symmetry reflects the geometric nature of the Nambu-Goto action: The area of the world sheet traced out by the string is independent of the coordinates (τ,σ) used to parametrize the world sheet. It is therefore a *local gauge* symmetry on the world sheet.
  - The transformation Eq. (15.19) marks X^μ as *scalar fields* on the world sheet (the Lorentz index only labels different fields!).
  - This symmetry follows from the invariance of the area element Eq. (15.16) under reparametrizations (coordinate transformations on the world sheet); cf. Eq. (10.101) for $D=2$.

iv | Define the following quantities (P^τ_μ are the canonical momenta of the field theory):

$$P^\tau_\mu := \frac{\partial L}{\partial \dot{X}^\mu} \overset{\circ}{=} -T\frac{(\dot{X}\cdot X')X'_\mu-(X')^2\dot{X}_\mu}{\sqrt{(\dot{X}\cdot X')^2-(\dot{X})^2(X')^2}} \tag{15.20a}$$

$$P^\sigma_\mu := \frac{\partial L}{\partial X'^\mu} \overset{\circ}{=} -T\frac{(\dot{X}\cdot X')\dot{X}_\mu-(\dot{X})^2X'_\mu}{\sqrt{(\dot{X}\cdot X')^2-(\dot{X})^2(X')^2}} \tag{15.20b}$$

→ Equation of motion (for arbitrary boundary conditions):

$$\frac{\partial P^\tau_\mu}{\partial \tau} + \frac{\partial P^\sigma_\mu}{\partial \sigma} = 0\,. \tag{15.21}$$

This is the equation of motion of a relativistic string on a D-dimensional Minkowski spacetime. It looks deceptively simple, but is actually extremely complicated due to Eq. (15.20). Luckily we will not have to solve it in this form.

v | Alternative action:

Similar to the point particle Eq. (15.8), The Nambu-Goto action is not well-suited for quantization due to the square root. We can find the analog of Eq. (15.11) by introducing an auxiliary field that makes the metric on the world sheet dynamical:

→ ⁂ *Polyakov action*:

$$\boxed{S_\mathrm{P}[h,X] := -\frac{T}{2}\int \underbrace{\sqrt{h}\,h^{ab}\partial_a X_\mu \partial_b X^\mu}_{=:\,L_\mathrm{P}/(-T/2)}\,\mathrm{d}\sigma\mathrm{d}\tau} \tag{15.22}$$

with dynamical world sheet metric h_{ab} ($a,b\in\{0,1\}$) and $h = |\det(h_{ab})|$.

- The Polyakov action (15.22) is classically equivalent to the Nambu-Goto action (15.17).

  To check this, calculate the EOMs for h^{ab} from Eq. (15.22) and plug them back into Eq. (15.22) to obtain Eq. (15.17).
- We denote the world sheet metric by h_{ab} (and not by g_{ab}) to emphasize that h_{ab} is *dynamical* and not necessarily the metric g_{ab} induced on the world sheet by the static spacetime.
- The metric h^{ab} has Lorentzian signature $(-,+)$.
- ¡! There are now two metric tensors involved: h_{ab} is the dynamical metric on the two-dimensional string world sheet, whereas $\eta_{\mu\nu}$ (hidden in the contraction of the μ-indices with $\mu = 0,\cdots,D-1$) is the static background metric of the D-dimensional spacetime on which the string propagates (here the Minkowski metric):

$$\partial_a X_\mu \partial^a X^\mu \equiv h^{ab}\partial_a X_\mu \partial_b X^\mu \equiv h^{ab}\eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu \tag{15.23}$$

World sheet metric (dynamic) Spacetime metric (static)

- The Polyakov action descibes D massless ← *Klein-Gordon fields* X^μ minimally coupled to the world sheet metric h_{ab}; recall Eq. (11.37) and remember that the fields X^μ are scalars, so that no covariant derivatives are needed (recall also ➔ Problemset 4).

vi | Symmetries of S_P:

- D-dimensional Poincaré invariance: (global & internal symmetry)

$$\bar{X}^\mu(\tau,\sigma) := \Lambda^\mu{}_\nu X^\nu(\tau,\sigma) + a^\mu \quad\text{and}\quad \bar{h}_{ab}(\tau,\sigma) := h_{ab}(\tau,\sigma) \tag{15.24}$$

  This symmetry is inherited from the Nambu-Goto action and reflects the fact that the integrand of the Polyakov action is still a spacetime scalar (and that Poincaré transformations are isometries of Minkowski space). Note that the world sheet metric transforms as a *scalar* under these transformations.

- Local gauge symmetries: (on the world sheet)
  - Diffeomorphism invariance:

a | Let us first define an alternative notation: $\sigma^0 := \tau$ and $\sigma^1 := \sigma$
A reparametrization/diffeomorphism can then be written in a compact form:

$$\left.\begin{aligned}\bar\tau &:= \bar\tau(\tau,\sigma)\\ \bar\sigma &:= \bar\sigma(\tau,\sigma)\end{aligned}\right\} \quad\Leftrightarrow\quad \bar\sigma^a := \bar\varphi^a(\sigma) \tag{15.25}$$

Here we use the shortcut $\sigma \equiv \{\sigma^0,\sigma^1\} = \{\tau,\sigma\}$.

The fields transform then as follows under diffeomorphisms:

$$\bar X^\mu(\bar\sigma) := X^\mu(\sigma) \qquad \text{(Scalar)} \tag{15.26a}$$

$$\bar h_{ab}(\bar\sigma) := \frac{\partial\sigma^c}{\partial\bar\sigma^a}\frac{\partial\sigma^d}{\partial\bar\sigma^b}h_{cd}(\sigma) \qquad \text{(Covariant rank-2 tensor)} \tag{15.26b}$$

Again, this reflects the fact that the parametrization of the string world sheet is unphysical and therefore a gauge symmetry.

¡! This transformation tells us that the D components X^μ are *scalar fields* on the world sheet. By contrast, they transform as *vector components* on Minkowski space (← *Poincaré symmetry*).

b | There is an important special class of diffeomorphisms:
$\bar\sigma^a = \varphi^a(\sigma)$ is a ⁂ *conformal diffeomorphism* (or ↑ *conformal map*) iff

$$\bar h_{ab}(\bar\sigma) = \frac{\partial\sigma^c}{\partial\bar\sigma^a}\frac{\partial\sigma^d}{\partial\bar\sigma^b}h_{cd}(\sigma) = \Omega(\sigma)\,h_{ab}(\sigma) \tag{15.27}$$

for some $\Omega(\sigma) > 0$.

* Conformal diffeomorphisms do not change angles. This is apparent from the rescaling of the metric tensor by $\Omega(\sigma)$, which only changes the length of tangent vectors, recall Eqs. (10.9) and (10.10).
* Conformal diffeomorphisms include ← *isometries* of the world sheet metric, i.e., coordinate transformations that do not change the components of the metric at all: $\Omega(\sigma) = 1$.

  - Weyl invariance:

The Polyakov action is invariant under the transformation

$$\tilde X^\mu(\tau,\sigma) := X^\mu(\tau,\sigma) \quad\text{and}\quad \underbrace{\tilde h_{ab}(\tau,\sigma) := e^{2\rho(\tau,\sigma)}h_{ab}(\tau,\sigma)}_{\text{⁂ Weyl transformation}} \tag{15.28}$$

for some real-valued function $\rho(\tau,\sigma)$.

* A transformation of this form is called ⁂ *Weyl transformation*.
* ¡! Note that Weyl transformations are active transformations of the world sheet metric, they are not coordinate transformations. Hence, Weyl transformations are not conformal diffeomorphisms.
* Use $\tilde h^{ab} = e^{-2\rho(\tau,\sigma)}h^{ab}$ to show the invariance of Eq. (15.22).

* The rescaling by $e^{2\rho}$ is a convenient way to make sure that the prefactor is positive for all functions ρ (which it must be for the new metric to remain regular everywhere).

To sum up:

- Conformal diffeomorphisms are a special class of diffeomorphisms that change the components of the world sheet metric only by a local factor (i.e., they act on points of the world sheet and move them around).
- Weyl transformations are a particular class of transformation of the *values* of the metric field by rescaling it locally *without moving points on the manifold around*.

Since the Polyakov action has both, Weyl invariance *and* diffeomorphism invariance …

$$\text{Diffeo. invariance:}\quad (h,X)\xrightarrow{\varphi}(\bar h,\bar X)\quad\Rightarrow\quad S_{\mathrm{P}}[\bar h,\bar X]=S_{\mathrm{P}}[h,X]\tag{15.29a}$$

$$\text{Weyl invariance:}\quad h\xrightarrow{\rho}\tilde h\quad\Rightarrow\quad S_{\mathrm{P}}[\tilde h,X]=S_{\mathrm{P}}[h,X]\tag{15.29b}$$

… we can combine them:

⊲ *Conformal* diffeomorphism φ →

$$S_{\mathrm{P}}[h,X]\overset{\text{Diff}}{=}S_{\mathrm{P}}[\bar h,\bar X]\overset{\text{Conf}}{=}S_{\mathrm{P}}[\tilde{\bar h},\bar X]\overset{\text{Weyl}}{=}S_{\mathrm{P}}[h,\bar X]\tag{15.30}$$

That is, we can use the Weyl symmetry to "undo" the effect of a conformal diffeomorphism on the metric, such that only the fields are affected by the conformal map. We call such transformations of the fields *conformal transformations*.

→ Conformal transformation φ is *symmetry* of S_P on *fixed* background metric h_{ab}.

→

$$\boxed{\text{The Polyakov action is a ⁂ \textit{conformal field theory}.}}\tag{15.31}$$

- Note that a particular class of conformal transformations are *global rescalings* of the world sheet: $(\tau,\sigma)\mapsto(\lambda\tau,\lambda\sigma)$; i.e., conformal field theories are *scale invariant*. This makes such theories (though not the Polyakov action) useful tools in condensed matter physics to describe second-order phase transitions (where systems become scale invariant due to fluctuations).
- The conformal symmetry will not survive the quantization of the Polyakov action in general; this is called ↑ *conformal/trace/Weyl anomaly*. Since the conformal symmetry is a (unphysical) *gauge* symmetry of the relativistic string, this poses a fundamental problem. The conformal symmetry can only be restored if (1) the spacetime dimension is $D=26$ and (2) the metric of *spacetime* satisfies the Einstein field equations.