↓Lecture 31 [05.08.24]

Epistemological disclaimer

In Part I and Part II we studied *widely accepted* and *experimentally tested* theories of nature: SPECIAL RELATIVITY and GENERAL RELATIVITY.

Here in Part III, we enter the realm of theories that are the brainchilds of theoretical physicists only – *without any experimental evidence supporting these theories*! We do not even know whether gravity is a quantum phenomenon to begin with …

Plan for this Excursion

This is a brief outlook on the fascinating but vast and complicated subject of *quantum gravity*; it is neither a comprehensive review nor a replacement for dedicated courses on the various subjects.

In this excursion, we address the following questions:

- Chapter 14:
  - *Why to quantize gravity in the first place?*
  - *How do we quantized non-gravitational theories?*
  - *Why does this procedure fail for gravity?*
  - *How to circumvent these problems?*
- Chapter 15:
  - *What is the rationale of string theory?*
  - *Why does the quantization of the bosonic string only work in $D = 26$ spacetime dimensions?*
  - *Why is string theory a theory of quantum gravity?*
  - *Where does supersymmetry enter the picture?*

Warning

The following conventions are widely used in the quantum gravity literature:

In this part …
…we work in units where $c = 1$ and $\hbar = 1$.
…we use the sign convention $\eta_{\mu\nu} = (-1, +1, \cdots, +1)$.

For example, the dispersion of a massive particle reads no longer $p^2 = m^2c^2$ but $p^2 = -m^2$.

14. Why is quantizing Gravity hard?

¡! Note that the Lorentz symmetry of SPECIAL RELATIVITY is not a problem for quantum theory (← Chapter 7). For example, the quantum field theories that constitute the Standard Model of particle physics are all Poincaré invariant and fully consistent with SPECIAL RELATIVITY. →

> The problem of quantum gravity is the quantization of the metric tensor field $g_{\mu\nu}$ of GENERAL RELATIVITY.

1 | *Why to quantize gravity in the first place?*

- Simple answer: *Because everything else can be quantized!*

$$\begin{aligned} \text{Quantum electrodynamics} &\xrightarrow{\hbar\to 0} \text{Maxwell's electrodynamics} \\ \text{Quantum mechanics} &\xrightarrow{\hbar\to 0} \text{Newton's classical mechanics} \\ \textit{What?} &\xrightarrow{\hbar\to 0} \text{Einstein's GENERAL RELATIVITY} \end{aligned}$$

  The fact that every classical theory – *except* GENERAL RELATIVITY– can be understood as the classical limit of an underlying quantum theory suggests that the superposition principle is a fundamental feature of reality, and motivates the quest for a quantum theory of the gravitational field (= the metric).

- Extrapolation of GENERAL RELATIVITY and quantum theory → Inconsistencies:

  i | Quantum mechanics:

  Heisenberg uncertainty: $\Delta x \Delta p \geq \frac{\hbar}{2} \to \langle p^2\rangle \geq (\Delta p)^2 \geq (\hbar/2\Delta x)^2$

  ◁ Relativistic particle: $E \sim cp \to E^2 \geq (\hbar c/2\Delta x)^2$

  In words: Probing small distances requires high energies (e.g., particle colliders).

  ii | GENERAL RELATIVITY:

  … Energy = Gravitational mass: $M \sim E/c^2$

  … Mass M concentrated in region $r_s = \frac{2GM}{c^2}$ → Black hole & Event horizon

  iii | Combing GENERAL RELATIVITY and quantum mechanics yields:

$$r_s \geq \frac{\hbar G}{\Delta x c^3} \tag{14.1}$$

  Imagine you want to mark a point in space with precision δl by placing a particle there. Then the particle must have position uncertainty $\Delta x \sim \delta l$. For $\delta l \to 0$, the particle requires more an more energy until a black hole forms and its event horizon prevents you from interacting with the particle. This happens when $\delta l \leq r_s$, i.e., latest when

$$\delta l \leq \frac{\hbar G}{\delta l c^3} \quad \Rightarrow \quad \delta l \leq l_{\text{Planck}} := \sqrt{\frac{\hbar G}{c^3}} \sim 10^{-35}\,\text{m}\,. \tag{14.2}$$

→ One cannot localize anything beyond the Planck length.

What this semiclassical argument shows is the impossibility to "zoom in" on the Planck scale and find a world in which both GENERAL RELATIVITY and quantum mechanics remain valid unmodified.

→

> The concept of space (and time) itself becomes inconsistent on the Planck scale l_{Planck}.

This argument goes back to MATVEI BRONSTEIN [300, 301]; he writes in 1936 in Ref. [301] (§4, p. 150):

> *Ohne eine tiefgehende Umarbeitung der klassischen Begriffe scheint es daher wohl kaum möglich, die Quantentheorie der Gravitation auch auf dieses Gebiet [der kleinen Abstände] auszudehnen.*

For a historical account on the early days of quantum gravity and the role played by Bronstein see Ref. [302].

- An argument from reductionism:

  Physics follows a reductionist approach to explain the world around us: All entities are split into smaller and smaller pieces that follow simpler and simpler laws (molecules → atoms → nuclei → quarks). The complexity of macroscopic phenomena is then explained as the *emergent* behavior of many simple constituents. This approach has worked remarkably well in compressing the apparent complexity of the world into a few simple fundamental laws.

  According to this view, the realm of the very small (studied by atomic and particle physics) is *fundamental*, everything else is *emergent*. But every single experiment that explored the realm of atomic or subatomic physics revealed a world goverened by the laws of quantum mechanics. There is no classical behaviour on subatomic scales! Thus, if we take the reductionist stance, we are forced to accept that *quantum mechanics rules the world*, and that our classical, macrospopic world is only an emergent perspective on this reality. Consequently, gravity should emerge from underlying quantum phenomena as well.

  The fly in the ointment is that no one has ever observed *any* effect of gravity – be it classical or quantum – in any experiment small enough to be clearly dominated by quantum effects *because gravity is such a weak force*: To see quantum effects, the studied systems must be extremely small (on atomic scales); but then the involved masses are also tiny. Since the gravitational coupling constant G is orders of magnitudes smaller than the electromagnetic coupling, every experiment on atomic scales is dominated by electromagnetic forces, while the gravitational force is practically absent.

  To date, the *smallest* object that showed measurable gravitational effects had a mass of $m \sim 0.5 \times 10^{-9}$ kg [303]. While this might seem light on everyday scales, it is still heavy on atomic scales: The *heaviest* object that showed quantum interference effects (↓ *double-slit experiment*) had a mass of $m \sim 4 \times 10^{-23}$ kg [304]. Because of decoherence (= coupling to the environment), it is experimentally extremely challenging to conduct experiments with relevant gravitational coupling while remaining coherent.

  [You might wonder: $m \sim 0.5 \times 10^{-9}\,\text{kg} = 0.5\,\text{mg}$ is quite "heavy". If one drops a particle of dust (which certainly weighs much less than 0.5 mg) in vacuum, it certainly falls to the ground. Doesn't this show that it interacts gravitationally? The answer is negative: This experiment only verifies the weak equivalence principle **WEP**, namely that everything – independent of its mass – is accelerated by g on the surface of Earth. The experiment reveals the spacetime curvature due to *Earth* by using the dust particle as a *test mass*. What is meant by "interacts gravitationally" is really "acts as a *source* of gravity," i.e., creates its own curvature of spacetime.]

- *Could gravity be intrinsically classical?*

  While it is certainly a majority view among physicists that gravity emerges from an underlying quantum theory, not everyone agrees on this. Roger Penrose, for example, advocates that "quantum mechanics must be gravitized." He denies that gravity has a quantum nature at all, and that the collapse of the wavefunction is an objective dynamical process – induced by gravity – that makes a unique, classical, macroscopic world emerge out of a microscopic quantum world [173]. This view is in direct contradiction to most other interpretations of quantum mechanics (collapse theories are not interpretations but *modifications* of quantum mechanics) like ↑ *Everett's many-worlds interpretation* or ↑ *decoherence theory*.

  Recent proposals suggest methods to experimentally probe the relation between gravity and the quantum-classical boundary [305–308] (see Ref. [309] for a review). These proposals are based on recent (and foreseeable) technological advances in the control of quantum systems and precision measurement techniques. Since there will be no experiments on the Planck scale anytime soon, this alternative approach to assess the quantum nature of gravity is perhaps the most promising route forward.

Ignoring the lack of experimental evidence, let us henceforth assume that gravity emerges from an underlying quantum theory.

2 | *How do we quantized non-gravitational theories?*

To understand why physicists struggle to quantize the field theory called GENERAL RELATIVITY, we must first understand how all the other fields are quantized:

Details: Any course on ↑ *Quantum field theory* [20].

i | ⊲ Relativistic field theory given by a Lagrangian:

$$\mathcal{L}(\phi,\partial\phi)=\underbrace{\overbrace{(\partial^\mu\phi)(\partial_\mu\phi)-m^2\phi^2}^{\mathcal{L}_0}}_{\substack{\text{Quadratic part}\\ \to\text{Free field}}}-\underbrace{\overbrace{\lambda\phi^{n\geq 4}+\dots}^{\mathcal{L}_{\text{int}}}}_{\substack{\text{Non-quadratic part}\\ \to\text{Interactions}}} \tag{14.3}$$

We use here exemparily a scalar field ϕ; the fields in the Standard Model are more complicated, but the concepts are the same.

→ Action:

$$S[\phi]=\int \mathrm{d}^dx\,\mathcal{L}(\phi,\partial\phi)\equiv S_0[\phi]+S_{\text{int}}[\phi] \tag{14.4}$$

For now, we consider arbitrary spacetime dimensions d.

So far, this defines a *classical* field theory with equations of motion $\delta_\phi S \overset{!}{=} 0$.

ii | The corresponding *quantum* theory is most conveniently defined via a ↓ *path integral*:

Define ↓ *Scattering amplitudes* via ↓ *path integrals*:

$$\mathcal{M} = \underbrace{\langle \phi_{\text{out}} | \phi_{\text{in}} \rangle}_{\substack{\text{Scattering}\\\text{amplitude}}} \sim \underbrace{\int_{\phi_{\text{in}}}^{\phi_{\text{out}}} \mathcal{D}\phi}_{\substack{\text{Sum over}\\\text{all evolutions}\\\text{(``Paths'')}}} \; \overbrace{e^{\frac{i}{\hbar} S[\phi]}}^{\substack{\text{Phase}\\\text{determined}\\\text{by action}}} \tag{14.5}$$

- You can think of the initial (final) field configuration ϕ_{in} (ϕ_{out}) as state that encodes the positions and momenta of many particles long before (after) they interact/collide. The scattering amplitude $\mathcal{M}$ is then the probability amplitude of this particular process happening. Such quantities can therefore be measured at particle colliders where such scattering experiment are performed.
- The path integral makes Feynman's interpretation of quantum mechanics explit, according to which all possible evolutions that connect an initial state ϕ_{in} with a final state ϕ_{out} happen simultaneously. The probability of the transition $\phi_{\text{in}} \mapsto \phi_{\text{out}}$ is then given by the (modulus square) of the sum of phases, each of which depends on the action of the particular path taken. Why? Because for $\hbar \to 0$ this construction yields as allowed transitions only the ones connected by trajectories that satisfy the classical equations of motion. The path integral therefore has the correspondence principle that connects quantum with classical physics built in.
- The benefit of the path integral approach over a canonical quantization via a Hamiltonian operator is that the former is based on the Lagrangian – which, for a relativistic field theory, is a Lorentz scalar. This makes the path integral quantization manifestly Lorentz covariant. By contrast, this symmetry is not manifest in a canonical quantization scheme, since the Hamiltonian is not a scalar but the zero-component of a 4-vector (the energy-momentum vector).

Problem:

> Without specifying the path integral, this is not a mathematically well-defined theory; it is only a (physically motivated) *sketch* of a theory.

This means that one must operationally define how exactly the "sum over all trajectories" is to be evaluated:

iii | How to compute the path integral Eq. (14.5)?

(1) It is hard to mathematically implement an integral over "all smooth functions ϕ". A first step is therefore to Fourier transform all fields and parametrize them by their Fourier components (a countable infinite set of real numbers). The path integral can then be performed by integrating over each of these Fourier components separately:

→ Fourier transform:

$$\left.\begin{matrix}\phi(x)\\ \mathcal{D}\phi\end{matrix}\right\} \quad\to\quad \left\{\begin{matrix}\hat{\phi}_k\\ \prod_k \mathrm{d}\hat{\phi}_k\end{matrix}\right. \tag{14.6}$$

(2) If $S_{\text{int}} \equiv 0$, the exponential includes only contributions from S_0 which are, by definition, *quadratic* in the fields (and the Fourier components). The integrals to be evaluated are then *Gaussian* and can be computed exactly. Such theories describe the propagation of particles that do not interact, hence they are called *free theories*. How the particles propagate is determined by their ↓ *propagator*, which can be directly computed from the Gaussian path integral and the free action S_0.

Interesting physics (= scattering) happens only when $S_{\text{int}} \neq 0$; in this case, the integrals are no longer Gaussian and one must resort to *perturbative methods* to find approximate solutions of the path integral:

→ Expand exponential in coupling parameter λ of interactions:

$$\mathcal{M} \sim \int_{\phi_{\text{in}}}^{\phi_{\text{out}}} \mathcal{D}\phi \underbrace{e^{\frac{i}{\hbar}S_0[\phi]}}_{\substack{\text{Gaussian}\\ \text{exponent}\\ \to \text{Propagator}}} \underbrace{\left[1 + \frac{i}{\hbar}S_{\text{int}} + \left(\frac{i}{\hbar}S_{\text{int}}\right)^2 + \dots\right]}_{\substack{\text{Perturbation expansion in } \lambda\\ \to \text{Interaction vertices}}} \tag{14.7}$$

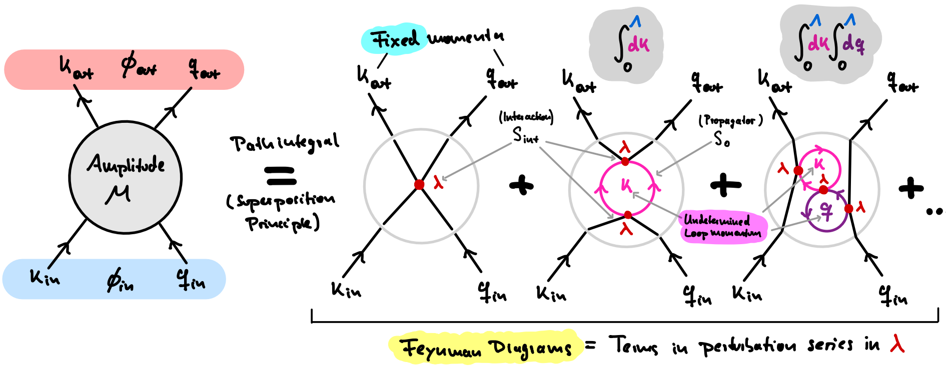
If the coupling constant $\lambda \propto S_{\text{int}}$ of the interaction is small, this approximation yields good results in low orders of the expansion.

This is true for quantum electrodynamics (QED), but not in the low-energy regime of quantum chromodynamics (QCD) which makes the latter much harder to work with.

In summary, the path integral can be evaluated as a perturbation series. These calculations are complicated, first because of the combinatorial problem to identify the different terms of the expansion that must be evaluated, and second, because evaluating the integrations associated to each of these terms is hard.

The first problem (writing down all terms up to a given order of the expansion) can be significantly simplified by using the technique of ↑ *Fenyman diagrams*:

→ Perturbation theory → Feynman diagrams:



- Each Feynman diagram can be translated via a dictionary of ↑ *Feynman rules* into a mathematical expression (typically including integrals) that must be evaluated. The infinite sum of all these expressions converges to the scattering amplitude.
- The amplitude is specified by the "legs" of the diagram: Say two particles with momenta k_{in} and q_{in} collide and scatter into two (potentially different) particles with momenta k_{out} and q_{out}. Quantum mechanics (via the path integral) tells us that the total amplitude of this process is the sum of the amplitudes of all possible processes consistent with these boundary conditions. The interaction Lagrangian $\mathcal{L}_{\text{int}}$ of the theory specifies the rules for allowed processes; these rules can be condensed into a set of ↑ *Feynman rules* specific for the theory. In general, each Feynman graph consists of *vertices* that

come from S_{int} and contribute one coupling constant λ to the overal expression. The links between the vertices (excluding the "legs" that stick out and are fixed by the boundary conditions) correspond to particles propagating between these interactions. Mathematically, each link corresponds to a ↓ *propagator* of the free theory.

iv | Momentum conservation demands that the sum of in- and outgoing momenta at each vertex of a Feynman diagram add up to zero. It is now easy to check that these constraints, together with the fixed external momenta $k_{\text{in}}, q_{\text{in}}, k_{\text{out}}, q_{\text{out}}$ do *not* fix the momenta of all propagators (links) if Feynman diagrams contain *loops*. This makes sense: the particle propagating along the loop can have *any* momentum without violating energy-momentum conservation at the vertices. Since we do not measure these particles, the path integral tells us that we must add up all possible values of these "loop momenta":

Path integral → Integrate over all *undetermined* momenta → *Loop* integrals

→ *Problem:* Divergent expressions for $k \to \infty$ in loops → UV-divergences

- Remember that large momenta $k \to \infty$ correspond to small distances and high energies; the limit $k \to \infty$ is therefore called ⁂ *UV-limit* and the corresponding divergences ⁂ *UV-divergences*.
- The occurence of these divergences is rather generic, and not specific to particularly "problematic" quantum field theories. One can interpret UV-divergences as indicators for their breakdown at very small distances (= high energies); i.e., quantum field theories are presumably *effective* descriptions of some other (UV-finite) theory that we do not know. In this regard, they are similar to the singularities of GENERAL RELATIVITY in that both can be interpreted as mathematical artifacts that signal the inconsistency (and thereby invalidity) of the theory in some domain.

The crucial question is whether these UV-divergencies make the whole endeavour (to describe particles by quantum field theories) a lost cause? After all, if computations yield only infinite results, we cannot make predictions about anything ☹ .

→ Temporary fix: Introduce *momentum cutoff* $\Lambda < \infty$ in all divergent integrals:

(This is called a ↑ *regularization*.)

$$\int_0^\infty \mathrm{d}k \quad \mapsto \quad \int_0^\Lambda \mathrm{d}k \tag{14.8}$$

This certainly removes all UV-divergencies and makes your results (scattering amplitudes) finite. The problem is that these now depend on the *unphysical* cutoff Λ, so that they cannot be measurable quantities anymore! We clearly didn't solve the problem but only masked it.

→ *Idea:*

Can we "hide" the terms that diverge for $\Lambda \to \infty$ in unphysical parameters?

The answer is "Yes" and the procedure is called ↑ *renormalization*. To get a feeling for the conditions that must be met for this to work, we must quantify the divergence of Feynman diagrams a bit more carefully:

v | Superficial degree of divergence:

The following line of arguments might seem sloppy; there are more rigorous derivations that come to the same conclusion but require more input from ↑ *quantum field theory* [20].

a | Recall: $\hbar = c = 1$ → Compton wavelength: $\lambda_c = \frac{h}{mc} = \frac{2\pi}{m}$
→ Dimension of length: $[\lambda_c] = M^{-1}$ (M: dimension of mass)

b | Dimension of action: $[S] = 1$ (since $\hbar = 1$)

c | $S = \int \mathrm{d}^d x\, \mathscr{L}$ and $[\mathrm{d}^d x] = M^{-d}$ → Dimension of Lagrangian: $[\mathscr{L}] = M^d$

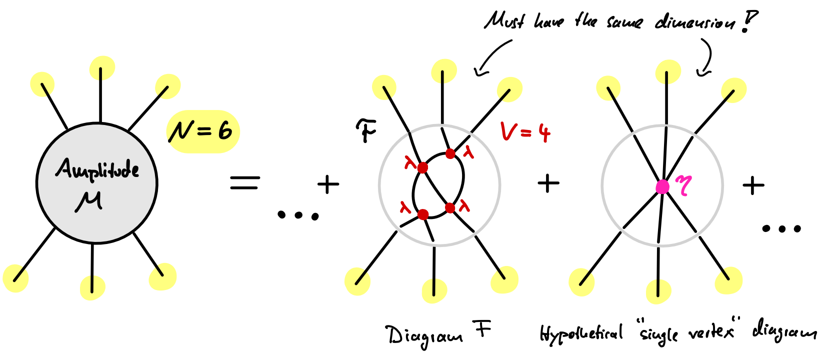
Since all dimensions can be expressed in M, we say that "$\mathscr{L}$ has (mass) dimension d".

d | From Eq. (14.3) follows (use $[\partial] = M$):

$$[\phi] = M^{\frac{d-2}{2}} \quad \text{and} \quad [\lambda] = M^{d-n\frac{d-2}{2}} \tag{14.9}$$

e | ⊲ Amplitude $\mathcal{F}$ of *single Feynman diagram* F with N external lines:

Has same dimension as (hypothetical) single interaction $\eta\phi^N \to [\eta] = M^{d-N\frac{d-2}{2}}$:



$\to [\mathcal{F}] = [\eta] = M^{d-N\frac{d-2}{2}}$

f | Let the Feynman diagram F have V interaction vertices →

$$\mathcal{F} \overset{\Lambda\to\infty}{\sim} \lambda^V \Lambda^D \tag{14.10}$$

D: ⁂ *Superficial degree of divergence* of F

After performing the integrals to compute the amplitude $\mathcal{F}$ from the Feynman diagram F, the only dimensionful quantities left are V powers of the coupling constant λ (one for each interaction vertex of the diagram) and D orders of the momentum cutoff Λ. In the limit $\Lambda \to \infty$, the asymptotic expression of $\mathcal{F}$ must therefore scale as $\lambda^V \Lambda^D$; note that this is an implicit *definition* of D. If $D > 0$, the contribution $\mathcal{F}$ has a UV-divergence.

→ (use $[\Lambda] = M$)

$$[\lambda]^V [\Lambda]^D = [\mathcal{F}] = M^{d-N\frac{d-2}{2}} \tag{14.11a}$$

$$\Rightarrow \quad V \log_M [\lambda] + D = d - N\frac{d-2}{2} \tag{14.11b}$$

$\log_M [\lambda]$: ⁂ *Mass dimension* of the coupling constant λ

→ We find for the superficial degree of divergence of $\mathcal{F}$:

$$D = d - \underbrace{\overbrace{\log_M [\lambda]}^{\substack{\text{Depends on} \\ \textit{diagram } F}}}_{\substack{\text{Eq. (14.9):} \\ d-n\frac{d-2}{2}}} \cdot V - \overbrace{\left(\frac{d-2}{2}\right) N}^{\substack{\text{Depends on} \\ \textit{amplitude } M}} \tag{14.12}$$

We can conclude for $d > 2$: ($D < 0 \Rightarrow$ Amplitude $\mathcal{F}$ converges for $\Lambda \to \infty$.)

- If the mass dimension of λ is zero or positive, diagrams with more "legs" (in- and out-going particles) become less divergent and eventually converge.
- Whether diagrams with more interaction vertices (= higher order of perturbation theory) start to converge or diverge depends on the *sign* of the mass dimension of the coupling constant.

g | → Classification:

This lead to the following classification of interacting quantum field theories:

- $\log_M[\lambda] > 0$ (Coupling constant has *positive* mass dimension.)

  → Only a finite number of *Feynman diagrams* (superficially) diverge ☺☺ .

  → ⁂ *Super-renormalizable theory*

  While this case is in some sense optimal, it is less relevant for interesting quantum field theories like the Standard Model; thus it plays no role in the following.

- $\log_M[\lambda] = 0$ (Coupling constant is *dimensionless*.)

  → Only a finite number of *amplitudes* (superficially) diverge ☺ .

  Here "amplitudes" refer to infinite sums of Feynman diagrams, classified by their number of external "legs" N.

  → ⁂ *Renormalizable theory*

  Most interesting quantum field theories (like the Standard Model) are of this type.

- $\log_M[\lambda] < 0$ (Coupling constant has *negative* mass dimension.)

  → All amplitudes diverge at sufficiently high order in perturbation theory ☹ .

  This follows because every amplitude has contributions from Feynman diagrams with arbitrary many vertices V so that – independent of N – the superficial degree of divergence D becomes positive for high-enough orders of perturbation theory.

  → ⁂ *Non-renormalizable theory*

h | Renormalization:

The following procedure of renormalization works (provably) for renormalizable (and super-renormalizable) theories because it assumes that a finite number of *amplitudes* are UV-divergent. One can then show rigorously that all UV-divergencies of such theories can be traced back to this finite set of divergent amplitudes (↑ *Weinberg theorem*). This implies that if these UV-divergences can be "cured", all scattering amplitudes of the theory become UV-finite. The procedure to "cure" a finite number of UV-divergent amplitudes is called *renormalization* and goes as follows:

(1) Start with a regularized (UV-cutoff Λ) (super-)renormalizable theory.

  → There is only a finite number of UV-divergent amplitudes.

(2) For each divergent amplitude, "add" a *counter term* with unphysical bare parameter to the Lagrangian.

  Strictly speaking you don't *add* the counterterms: You *split* the terms with bare parameters into (fixed) physical parameters and (UV-divergent) unphysical parameters; the latter are the counter terms.

(3) One can then show that all UV-divergences can be absorbed by these (unobservable & unphysical) *bare* parameters by fixing their corresponding (observable & physical) *renormalized* parameters.

(4) You end up with a theory that yields finite scattering amplitudes in the limit $\Lambda \to \infty$ and reproduces the observed physical parameters every order of perturbation theory ☺ .

The UV-divergencies are now "hidden" in the *bare* parameters and make them diverge in the limit $\Lambda \to \infty$. But this is not a problem because they do not affecet observable quantitites.

i | This means in particular:

> Renormalizable quantum field theories allow for the computation of predictions by fixing a finite number of physical low-energy parameters.

- For QED this would be the physical electron mass m and charge e; in the Standard Model there are about 18 such parameters that determine the masses and interactions of elementary particles. These must be *measured* and can then be used to make predictions about scattering amplitudes. This is why the Standard Model cannot *predict* the masses of elementary particles (e.g., the Higgs boson).
- You might wonder: If we must use masses and interaction strengths of particles as *input* of our theories, what is it actually good for? Remember that the predictions of quantum field theories are *scattering amplitudes* $\mathcal{M}$. These are complicated functions of the momenta (both absolute value and direction) of the in- and outgoing particles. It is this highly non-trivial functional form that is predicted by the theory – which can be compared to scattering experiments. In the case of the Standard Model, theory and experiment match perfectly!
- Here is a very accessible explanation of renormalization by John Baez:

  `https://math.ucr.edu/home/baez/renormalization.html`

But conversely:

> We do not know how to define and/or extract predictions from non-renormalizable quantum field theories.

For a non-renormalizable QFT we would have to add infinitely many counter terms and fix infinitely many physical parameters to absorb the infinitely many UV-divergent amplitudes. This makes such theories useseless and conceptually ill-defined.

vi | Examples:

- $\sphericalangle$ Scalar field Eq. (14.3) in $d = 4$ with $n = 4$:

$$\log_M[\lambda] \overset{14.9}{=} d - n\frac{d-2}{2} = 0 \tag{14.13}$$

$\to \phi^4$-theory is renormalizable in $d = 3+1$ spacetime dimensions.

- $\sphericalangle$ Quantum electrodynamics (QED) in $d = 4$:

$$\mathcal{L}_{\text{QED}}(A, \partial A, \Psi, \partial\Psi) = \overbrace{\underbrace{\bar{\Psi}(i\partial\!\!\!/ - m)\Psi}_{\text{Free fermions}} - \underbrace{\frac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\text{Free photons}}}^{\mathcal{L}_0} \overbrace{- e\bar{\Psi}\gamma^\mu\Psi A_\mu}^{\mathcal{L}_{\text{int}}} \tag{14.14}$$

e: Coupling constant (= electric charge of fermion Ψ)

→ Dimensional analysis:

$$[A] \mathrel{\hat{=}} M^{\frac{d-2}{2}} = M^1\,,\quad [\Psi] \mathrel{\hat{=}} M^{\frac{d-1}{2}} = M^{3/2} \quad \xRightarrow{14.14} \quad \log_M [e] \mathrel{\hat{=}} 0 \tag{14.15}$$

→ QED is (superficially) renormalizable in $d = 3+1$ spacetime dimensions ☺ .

- Note that we only showed that QED is *superficially* renormalizable by essentially dimensional anlysis (= power counting). This is not a rigorous proof that QED really is renormalizable to all orders of perturbation theory – it is only suggestive that it might be. However, one can show rigorously that QED is renormalizable to all orders of perturbation theory, although such proofs are very technical [310].
- The same is true for the strong interactions of quantum chromodynamics (QCD) and the electroweak interactions: One (more precisely: GERARD 'T HOOFT) can prove (to the standards of theoretical physicists) that the full Standard Model is renormalizable to all orders of perturbation theory [311, 312]. This yields operationally well-defined quantum field theories for three of the four fundamental forces of nature:
  * Electromagnetic force ✓
  * Weak force ✓
  * Strong force ✓

3 | *Why does this procedure fail for gravity?*

After this preliminary work, the question to answer is clear:

Is GENERAL RELATIVITY*– defined by the Einstein-Hilbert action – renormalizable?*

i | ⊲ Pure gravity → Einstein-Hilbert action Eq. (12.54) with $\kappa := \sqrt{16\pi G}$:

$$S_{\text{EH}}[g] = \frac{1}{\kappa^2}\int \mathrm{d}^4x\,\sqrt{g}\,R \tag{14.16}$$

Problem: This is not in the form $S_0 + S_{\text{int}}$ required for perturbation theory!

This means that we cannot simply declare the gravitational constant $\kappa^2 \propto G$ as the coupling parameter and draw conclusions from its mass dimension.

Note that due to the non-linearity of the Einstein field equations, pure gravity (without matter) is already an interacting field theory that must be solved perturvatively.

ii | Expand Eq. (14.16) around static background spacetime:

$$g_{\mu\nu} \equiv \underbrace{\eta_{\mu\nu}}_{\text{Background}} + \underbrace{\kappa h_{\mu\nu}}_{\substack{\text{Quantum}\\\text{fluctuations}}} \tag{14.17}$$

The choice to rescale the field by κ is covenient to bring the action → *below* into the standard form needed for perturbation theory.

Eq. (14.17) →

$$\sqrt{g} \mathrel{\hat{=}} 1 + \tfrac{\kappa}{2}h_\mu{}^\mu + \tfrac{\kappa^2}{8}h_\mu{}^\mu h_\nu{}^\nu - \tfrac{\kappa^2}{4}h_{\mu\nu}h^{\mu\nu} + \mathcal{O}(h^3) \tag{14.18a}$$

$$R \mathrel{\hat{=}} \kappa\partial^2 h^\mu{}_\mu - \kappa\partial_\mu\partial_\nu h^{\mu\nu} + \mathcal{O}(h^2) \tag{14.18b}$$

Expanding the Einstein-Hilbert action in the fluctuations $h_{\mu\nu}$ yields:

$$S_{\mathrm{EH}}[h] \stackrel{\circ}{=} \int \mathrm{d}^4x \Big[\underbrace{\left\{ \begin{matrix} \frac{1}{2}\partial^\nu h_{\mu\nu}\partial_\sigma h^{\mu\sigma} - \frac{1}{2}\partial^\nu h_{\mu\nu}\partial^\mu h_\sigma{}^\sigma \\ +\frac{1}{4}\partial^\mu h_\nu{}^\nu \partial_\mu h_\sigma{}^\sigma - \frac{1}{4}\partial_\sigma h_{\mu\nu}\partial^\sigma h^{\mu\nu} \end{matrix} \right\}}_{\mathcal{L}_0 \,\to\, \text{Graviton propagator}} + \underbrace{\kappa(\partial h)^2 h + \dots}_{\mathcal{L}_{\mathrm{int}} \,\to\, \text{Interactions}} \Big] \tag{14.19}$$

- Note that the κ^2 cancels in the quadratic terms, while coupling constants survive in the higher-order interaction terms (this is why we rescaled the field in the first place).
- $\mathcal{L}_0$ should be familiar: You studied this theory on ➔ Problemset 1 as a first attempt at a relativistic theory of gravity.
- Here we only write one of the lowest-order interaction terms exemplarily (omitting indices); it is useful to derive the mass dimension of κ from the mass dimension of $h_{\mu\nu}$ (which, in turn, is fixed by the non-interacting quadratic part). That an interaction term of this form exists follows from Eq. (14.18) via partial integration; see Ref. [313] for details.

iii | In principle you can start now to derive the Feynman rules from Eq. (14.19) to compute scattering amplitudes of the Einstein-Hilbert quantum gravity.

When evaluating the path integral (e.g., to compute the propagator), a complication arises: Eq. (14.19) is a *gauge theory* [due to the diffeomorphism invariance of Eq. (14.16); ➔ Problemset 6 and ← Section 13.4 and also Eq. (11.103)]. If one naïvely calculates the path integral of a gauge theory, all expressions blow up because the gauge orbits don't oscillate and produce infinities. To count physically distinct field configuration only once, one has to add a gauge-fixing term to the Lagrangian. In doing so, one encounters a functional determinant that leads to new, artificial fields called ↑ *Fadeev-Popov ghosts*. They are a necessary mathematical nuisance and expand the list of Feynman rules & diagrams. Because of this, the interactions of the Einstein-Hilbert action, and the fact that $h_{\mu\nu}$ is a rank-2 tensor field, enumerating and evaluating Feynman diagrams of this theory is not fun (even ignoring potential UV-divergences).

By fixing the gauge appropriately, one can compute the graviton ↑ *Feynman propagator* from the quadratic part $\mathcal{L}_0$ of Eq. (14.19) (↑ Ref. [313]) and finds

$$D^F_{\mu\nu\alpha\beta}(k) \stackrel{*}{=} \frac{\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta}}{k^2 + i\varepsilon} \,. \tag{14.20}$$

This is nice but eventually futile because the theory contains infinitely many UV-divergencies that one cannot control (→ *next*).

iv | In Eq. (14.19), κ plays the role of the coupling constant that controls graviton-graviton scattering. Hence its mass dimension determines the renormalizability of GENERAL RELATIVITY:

Dimensional analysis of Eq. (14.19):

$$[h] \stackrel{\circ}{=} M^{\frac{d-2}{2}} = M^1 \quad \xrightarrow{14.19} \quad \log_M[\kappa] = -1 \;< 0$$

→ κ has negative mass dimension!

Here is a sanity check: The gravitational constant can be written as $G = \hbar c/m^2_{\mathrm{Planck}}$ with Planck *mass* m_{Planck}. With $\hbar = 1 = c$ it follows that G has dimensions of M^{-2}, consistent with our result above for $\kappa = \sqrt{16\pi G}$.

v | Conclusion:

> GENERAL RELATIVITY is superficially *non-renormalizable* ☹ .

¡! This does not *prove* that the Einstein-Hilbert action is not renormalizable; but if it were, some sort of unexpected cancellations/symmetries would be necessary (→ *next*).

vi | Known results:

- One can show that at one-loop level, *pure* Einstein gravity (no matter fields) has – quite unexpectedly! – no UV-divergences [314].
- However, when matter is involved, the one-loop diagrams of the Einstein-Hilbert action become UV-divergent, see Ref. [314] for the example of a scalar field (see also references in Ref. [315]).
- Unfortunately, pure Einstein gravity is proven to be UV-divergent at two-loop level [315, 316]. This suggests that no unexpected cancellations/symmetries make the theory renormalizable.
- It is therefore widely believed (though, to my knowledge, not proven) that no unexpected cancellations occur beyond two-loop order; therefore, Einstein gravity seems to be perturbatively non-renormalizable.
- For an alternative (and pedagogic) explanation for the non-renormalizability of GENERAL RELATIVITY see Ref. [317].

vii | In a nutshell:

- GENERAL RELATIVITY is different from the field theories of the Standard Model in that its coupling constant has *negative* mass dimension.
- As a consequence, the only systematic procedure to operationally define quantum field theories (namely: renormalization) does *not* work for GENERAL RELATIVITY.
- However, there is no rigorous proof that GENERAL RELATIVITY *cannot* be quantized by another (non-perturbative?) method.

4 | *How to circumvent this problem?*

Since the conventional method to study quantum field theories fails for GENERAL RELATIVITY, one needs new methods to tackle the problem. There are two very different approaches:

- Approach 1:

> Try to "rediscover" GENERAL RELATIVITY
> in some limit of a UV-finite quantum field theory.

Most prominent contender: → *String theory* (Chapter 15)

String theory does not only claim to provide a path for quantizing the gravitational field, but also strives to explain the existence and interactions of all other particles ("matter") in a single, consistent framework. Its aspiration is therefore not only to be a theory of quantum gravity but a "theory of everything" (ToE). The emergence of GENERAL RELATIVITY is only one of its aspects.

- Approach 2:

> Try to come up with an alternative (non-perturbative?) method to quantize the metric field of GENERAL RELATIVITY.

Most prominent contender: ↑ *Quantum loop gravity*

Quantum loop gravity "takes GENERAL RELATIVITY seriously" and directly tries to quantize the theory by discretizing its degrees of freedom and proposing a UV-finite action & path integral that determine the dynamics of the geometry of spacetime. Quantum loop gravity (in its basic incarnation) does not contain matter fields; it is "just" a theory of quantum gravity. In contrast to string theory, quantum loop gravity does not claim to be a "theory of everything".