

↓ **Lecture 3** [31.10.23]

c | Active symmetries:

There is something additional and particularly useful to be learned from the coordinate transformation above. We showed:

$$\text{If } \vec{X}_i(t) \text{ satisfies } m_i \frac{d^2 \vec{X}_i(t)}{dt^2} = \sum_{k \neq i} \vec{F}_{k \rightarrow i}(\vec{X}_k(t) - \vec{X}_i(t)) \quad (1.28a)$$

$$\text{then } \vec{X}'_i(t') \text{ satisfies } m_i \frac{d^2 \vec{X}'_i(t')}{dt'^2} = \sum_{k \neq i} \vec{F}_{k \rightarrow i}(\vec{X}'_k(t') - \vec{X}'_i(t')) \quad (1.28b)$$

But t' in the lower statement is just a dummy variable that can be renamed to whatever we want:

$$\text{If } \vec{X}_i(t) \text{ satisfies } m_i \frac{d^2 \vec{X}_i(t)}{dt^2} = \sum_{k \neq i} \vec{F}_{k \rightarrow i}(\vec{X}_k(t) - \vec{X}_i(t)) \quad (1.29a)$$

$$\text{then } \vec{X}'_i(t) \text{ satisfies } m_i \frac{d^2 \vec{X}'_i(t)}{dt^2} = \sum_{k \neq i} \vec{F}_{k \rightarrow i}(\vec{X}'_k(t) - \vec{X}'_i(t)) \quad (1.29b)$$

Use colors to highlight the changes.

→ $\vec{X}'_i(t) = R\vec{X}_i(t - s) + \vec{v}(t - s) + \vec{b}$ is a *new solution* of Eq. (1.16)!

Note that for $s = 0$ it is $\vec{X}'_i(0) = R\vec{X}_i(0) + \vec{b}$ and $\dot{\vec{X}}'_i(0) = R\dot{\vec{X}}_i(0) + \vec{v}$, i.e., the solution $\vec{X}'_i(t)$ satisfies different initial conditions.

→ We say:

The Galilei group \mathcal{G} is an \ast *invariance group* or an (*active*) *symmetry* of Eq. (1.16).

↓ **Interlude: Active and passive transformations**

It is important to understand the conceptual difference between the two last points:

- In the previous step we took a specific trajectory (solution of Newton's equation) and expressed it in different coordinates. We then found that the differential equation obeyed by *the same physical trajectory* in these new coordinates “looks the same” as in the old coordinates. We called this peculiar feature of the differential equation “Galilei-covariance” or “form-invariance”. This type of a transformation is called *passive* because we keep the physics the same and only change our description of it.
- In the last step, we have shown that there is a dual interpretation to this: If a differential equation is form-invariant under a coordinate transformation, then we can exploit this fact to construct *new solutions* from given solutions (in the same coordinate system!). This type of transformation is called *active* because we keep the coordinate frame fixed and actually *change the physics*. You can therefore think of active transformations/symmetries as “algorithms” to construct new solutions of a differential equation (a quite useful feature since solving differential equations is often tedious).

10 | Galilean relativity:

i | **Remember:**

The law of inertia holds (by definition) in all inertial systems.

→ The “inertial test” **IN** cannot be used to distinguish inertial systems.

This is a tautological statement because we *define* inertial systems in this way!

Empirical fact:

Every mechanical experiment (not just the “inertial test”) yields the same result in all inertial systems.

This is not a tautology but an empirically tested feature of reality.

This motivates the following postulate (first given by Galileo Galilei):

§ Postulate: Galilei’s principle of Relativity **GR**

No mechanical experiment can distinguish between inertial systems.

! In this formulation, **GR** encodes a (so far uncontested) empirical fact. In particular, it does neither refer nor rely on (the validity of) any *physical model*, e.g., Newtonian mechanics. As such we should expect that it survives our transition to **SPECIAL RELATIVITY**.

Here is a more operational formulation of **GR**: You describe a detailed experimental procedure using equipment governed by mechanics (springs, pendula, masses, ...) that can be performed in a closed (but otherwise perfectly equipped) laboratory. Then you copy these instructions without modifications and hand them to scientists with labs in different inertial systems. They all perform your instructions and get some results (e.g. the final velocities of a complicated contraption of pendula). When they report back to you, their results will all be identical. This is the essence of **GR**.

ii | In the language of *models* that describe the mechanical laws faithfully, **GR** can be reformulated:

§ Postulate: Galilei’s principle of Relativity **GR'**

The equations that describe mechanical phenomena *faithfully* have the same form in all inertial systems.

If this would not be the case you could distinguish between different inertial systems by checking which formula you have to use to describe your observations. Imagine a rotating (non-inertial) frame where you have to use a modified version of Newton’s EOMs (that include additional terms for the Coriolis force) to describe your observations.

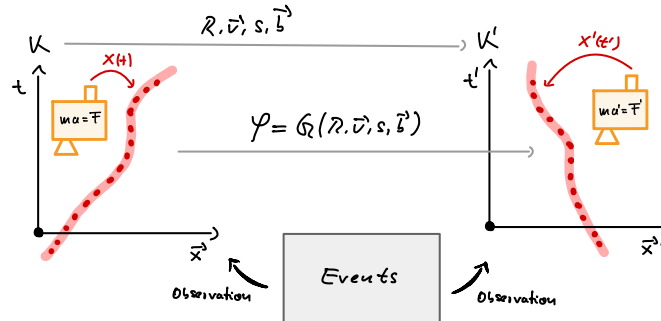
Note that “the same form” actually means that the models are *functionally equivalent* (have the same solution space). Functional equivalence is equivalent to the *possibility* to formulate the model (= equation of motion) in the same form.

iii | Under the assumption (!) that Newtonian physics (in particular Eq. (1.16)) describes mechanical phenomena *faithfully*, this implies:

Newton’s equations of motion have the same form in all inertial systems.

! This statement is *not* equivalent to **GR** or **GR'** as it relies on an independent empirical claim (namely the validity of Newton's equation as a model of mechanical phenomena).

We can now combine this claim with our (purely mathematical!) finding concerning the invariance group of Newton's equations:



→ Preliminary/Historical conclusion:

$$\varphi(K \xrightarrow{R, \vec{v}, s, \vec{b}} K') \stackrel{?}{=} G(R^{-1}, -\vec{v}, -s, -\vec{b}) \in \mathcal{G}$$

Recall that rotating the coordinate *axes* by R makes the coordinates of fixed events rotate in the *opposite* direction R^{-1} ; the same is true for the other transformations.

Since this is a course on RELATIVITY, we should be skeptical (like Einstein) and ask:

Is this true?

1.3. Einstein's principle of special relativity

11 | Mathematical fact:

The Maxwell equations of electrodynamics are *not* Galilei-covariant.

Proof: → Problemset 1

Here for your (and my) convenience the Maxwell equations in vacuum (in cgs units):

$$\text{Gauss's law (electric): } \nabla \cdot \mathbf{E} = 0 \tag{1.30a}$$

$$\text{Gauss's law (magnetic): } \nabla \cdot \mathbf{B} = 0 \tag{1.30b}$$

$$\text{Law of induction: } \nabla \times \mathbf{E} = -\frac{1}{c} \partial_t \mathbf{B} \tag{1.30c}$$

$$\text{Ampère's circuital law: } \nabla \times \mathbf{B} = \frac{1}{c} \partial_t \mathbf{E} \tag{1.30d}$$

“Handwavy explanation” for the absence of Galilei symmetry:

The Maxwell equations imply the wave equation for both fields:

$$\left(\nabla^2 - \frac{1}{c^2} \partial_t^2 \right) \mathbf{X} = \mathbf{0} \quad \text{for } \mathbf{X} \in \{\mathbf{E}, \mathbf{B}\}. \tag{1.31}$$

Here the speed of light c plays the role of the phase and group velocity of the waves; i.e., all light signals propagate with c . Form-invariance under some coordinate transformation φ implies that the same light signal propagates with *the same velocity* c in all coordinate systems related by φ . This is clearly incompatible with the Galilean law for adding velocities (according to which a signal with velocity u'_x in frame K' propagates with velocity $u_x = u'_x + v_x$ in frame K if $K \xrightarrow{v_x} K'$).

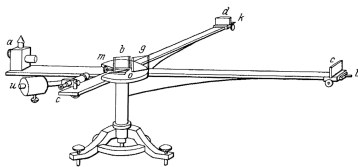
12 | The simplest escape from our predicament:

Maybe there is no relativity principle for electrodynamics?

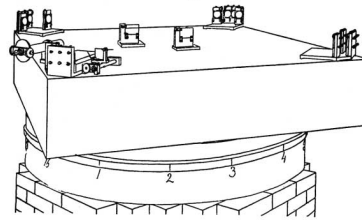
Reasoning: If we cling to the validity of Newtonian mechanics and Galilean relativity **GR**, we are forced to assume $\varphi = G$ as the transformation between inertial systems. Since the Maxwell equations are *not* form-invariant under these transformations, they look differently in different inertial systems. So there must be a (class of) designated inertial coordinate systems $[K_0]$ in which the Maxwell equations in the specific form Eq. (1.30) you've learned in your electrodynamics course are valid.

→ $[K_0]$ = Frame in which the “*luminiferous aether*” is at rest (?)

13 | Michelson Morley experiment (plots from [26, 27]):



Michelson's original setup (1881)



Michelson & Morley's improved setup (1887)

→ The speed of light is the same in all directions.

→ There is no “*luminiferous aether*” $[K_0]$.

(Or it is pulled along by earth – which contradicts the observed \uparrow *aberration of light*.)

→ The speed of light c *cannot* be fixed wrt. some designated reference frame $[K_0]$.

→ *No experimental evidence* that the Maxwell equations *do not hold* in all inertial systems.

→ *Relativity principle* for electrodynamics?!

- Historical note:

A. Einstein writes in a letter to F. G. Davenport (see Ref. [28]):

[...] In my own development Michelson's result has not had a considerable influence. I even do not remember if I knew of it at all when I wrote my first paper on the subject (1905). The explanation is that I was, for general reasons, firmly convinced how this could be reconciled with our knowledge of electro-dynamics. One can therefore understand why in my personal struggle Michelson's experiment played no role or at least no decisive role.

→ The Michelson Morley experiment did *not* kickstart SPECIAL RELATIVITY.

- Modern Michelson-Morley like tests of the isotropy of the speed of light achieve much higher precision than the original experiment. The authors of Refs. [29, 30], for example, report an upper bound of $\Delta c/c \sim 10^{-17}$ on potential anisotropies of the speed of light by rotating optical resonators.

14 | Two observations:

- (1) *No* evidence that there is *no* relativity principle for electrodynamics.

(2) Why does Galilean relativity GR treat mechanics differently anyway?

Put differently: Why should mechanics, a branch of physics artificially created by human society, be different from any other branch of physics? This is not impossible, of course, but it certainly lacks simplicity! (To Galilei’s defence: At his time “mechanics” was more or less identical to “physics”.)

→ A. Einstein writes in §2 of Ref. [9] as his first postulate:

1. Die Gesetze, nach denen sich die Zustände der physikalischen Systeme ändern, sind unabhängig davon, auf welches von zwei relativ zueinander in gleichförmiger Translationsbewegung befindlichen Koordinatensystemen diese Zustandsänderungen bezogen werden.

We reformulate this into the following postulate:

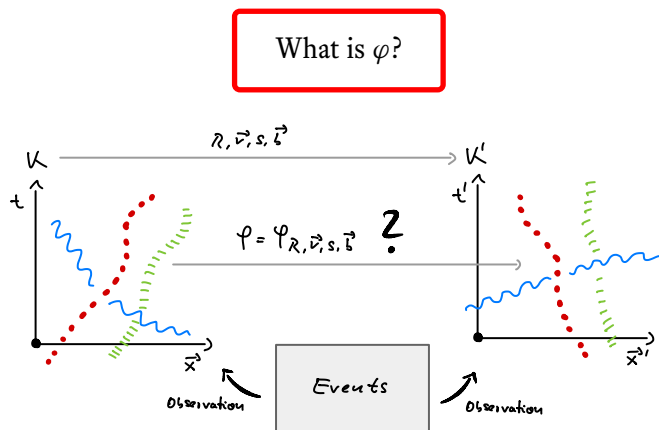
§ Postulate: (Einstein’s principle of) Special Relativity SR

No ~~mechanical~~ experiment can distinguish between inertial systems.

Note the difference to Galilean relativity GR according to which no experiment governed by classical mechanics can distinguish between inertial systems. Einstein simply extended this idea to all of physics – no special treatment for mechanics!

! There are various names used in the literature to refer to SR. Here we call it the principle of special relativity, where the “special” refers to its restriction on inertial systems – as compared to the principle of general relativity in GENERAL RELATIVITY that refers to all frames (→ later). To emphasize its difference to Galilean relativity GR, some authors call SR the universal principle of relativity, where “universal” refers to its applicability on all laws of nature (not just the realm of classical mechanics).

- 15 | But now that there are more contenders (mechanics, electrodynamics, quantum mechanics) all of which must be invariant under the same transformation φ , we have to open the quest for φ again:



The differently colored/shaped trajectories symbolize phenomena of mechanics (red), electrodynamics (blue), and quantum mechanics (green). According to SR, all of them must be form-invariant under a common coordinate transformation φ .

! To reiterate: This is not a question about symmetry properties of equations or models! It is an experimentally testable fact about reality. There is only one correct φ and it is just as real as the three-dimensionality of space.

1.4. Transformations consistent with the relativity principle

Since this is a theory lecture, so we cannot do experiments. Let us therefore weaken the question slightly:

What is most general form of φ consistent with reasonable assumptions about reality?

§ Assumptions

- SR** *Special Relativity*: There is no distinguished inertial system.
- IS** *Isotropy*: There is no distinguished direction in space.
- HO** *Homogeneity*: There is no distinguished place in space or point in time.
- CO** *Continuity*: φ is a continuous function (in the origin).

Something is “distinguished” if there exists an experiment that can be used to identify it unambiguously.

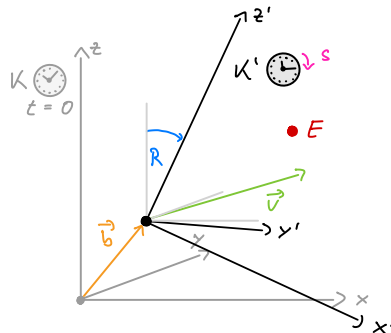
This derivation follows Straumann [8] with input from Schröder [1] and Pal [31].

Detailed calculations: → Problemset 2

1 | Setup:

◁ Two inertial systems $K \xrightarrow{R, \vec{v}, s, \vec{b}} K'$.

◁ Event $E \in \mathcal{E}$ with coordinates $x \equiv (t, \vec{x})_K \in E$ and $x' \equiv (t', \vec{x}')_{K'} \in E$:



We are interested in the transformation $\varphi \equiv \varphi_{R, \vec{v}, s, \vec{b}}$ with

$$x' = \varphi(x). \tag{1.32}$$

Note that **SR** forbids us to use the inertial system labels K or K' in the definition of φ ! We can only use the relative parameters (R, \vec{v}, s, \vec{b}) measured in K wrt K' .

2 | Affine structure:

Our first goal is to show that φ must be an affine map.

- i | ◁ Event $\tilde{E} \in \mathcal{E}$ with coordinates $\tilde{x} = x + a$ in K for some shift $a \in \mathbb{R}^4$.
- ii | Homogeneity **HO** →

$$\varphi(x + a) - \varphi(x) \stackrel{!}{=} a'(\varphi, a) \tag{1.33}$$

$a'(\varphi, a)$: Shift in K' independent of x (this reflects homogeneity in space and time)

Imagine the right-hand side $a'(\varphi, a)$ where *not* independent of x . Then there would be an interval (say, a rod of spatial extend \vec{a}) that has the same length \vec{a} in K no matter where it is located, but *variable* length $\vec{a}(\varphi, \vec{a}, \vec{x})$ in K' as a function of \vec{x} . The observer in K' can then use this “magic rod” to pinpoint absolute positions in space (the same argument works in time, then with a clock instead of a rod).

iii | For $x = 0$: $a'(\varphi, a) = \varphi(a) - \varphi(0) \rightarrow$

$$\varphi(x + a) = \varphi(x) + \varphi(a) - \varphi(0). \quad (1.34)$$

iv | Let $\Psi(x) := \varphi(x) - \varphi(0) \rightarrow$

$$\Psi(x + a) = \Psi(x) + \Psi(a) \quad \text{and} \quad \Psi(0) = 0. \quad (1.35)$$

This would be satisfied if Ψ were *linear*! But we do not know this yet ...

v | **Claim:** $\Psi(x)$ continuous at $x = 0$ (follows from **co**) $\Rightarrow \Psi$ is *linear*.

a | Eq. (1.35) $\rightarrow \Psi(nx) = n\Psi(x)$ for $n \in \mathbb{N}$ (show by induction!)

b | Eq. (1.35) $\rightarrow \Psi(-x) = -\Psi(x)$ (use $\Psi(0) = 0$) $\rightarrow \Psi(nx) = n\Psi(x)$ for $n \in \mathbb{Z}$

c | \triangleleft Rational number $r = \frac{m}{n}$, $m, n \in \mathbb{Z} \rightarrow$

$$r\Psi(x) = \frac{m}{n}\Psi(x) = \frac{1}{n}\Psi(mx) = \frac{1}{n}\Psi(nrx) = \frac{n}{n}\Psi(rx) = \Psi(rx). \quad (1.36)$$

d | $\Psi(x)$ continuous at $x = 0 \xrightarrow{\text{Eq. (1.35)}} \Psi(x)$ continuous everywhere.

Show this using the definition of continuity, i.e., $\lim_{x \rightarrow 0} \Psi(x) = \Psi(0)$!

e | $r\Psi(x) = \Psi(rx)$ for $r \in \mathbb{Q} \xrightarrow{\Psi \text{ continuous}} r\Psi(x) = \Psi(rx)$ for $r \in \mathbb{R}$

Remember that real numbers are defined in terms of (equivalence classes of) limits of rational numbers, i.e., \mathbb{Q} is dense in \mathbb{R} .

f | **In conclusion:**

$$\Psi(x + a) = \Psi(x) + \Psi(a) \quad \text{and} \quad \Psi(rx) = r\Psi(x) \quad (1.37)$$

$\rightarrow \Psi$ is *linear*.

vi | If Ψ is linear, $\varphi(x) = \Psi(x) + \varphi(0)$ is *affine*:

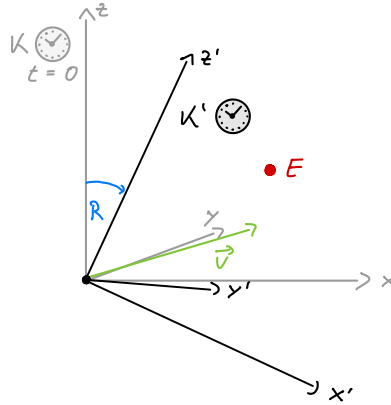
$$\varphi(x) = \Lambda x + a \quad (1.38)$$

with $\Lambda = \Lambda(R, \vec{v}, s, \vec{b})$ a 4×4 matrix and $a = a(R, \vec{v}, s, \vec{b})$ a 4-dimensional vector.

3 | The spacetime translation a is simply $a = (-s, -\vec{b})$ [recall Eqs. (1.7) and (1.9)].

$\rightarrow \triangleleft$ Homogeneous transformations ($a = 0$) in the following:

$$x' = \varphi(x) = \Lambda x. \quad (1.39)$$



- 4 | We already know from our discussion of inertial systems [recall Eq. (1.11)]:
Rotation group SO(3) must be part of the transformations φ with representation

$$x' = \Lambda_{R^{-1}} x \quad \text{with} \quad \Lambda_R := \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & R \end{array} \right) \quad \text{where} \quad R \in \text{SO}(3). \quad (1.40)$$

This is just a fancy way to rewrite Eq. (1.11).

- 5 | **** Pure boost** $K \xrightarrow{1, \vec{v}, 0, \vec{0}} K'$:

- i | $\triangleleft (t)_K = 0 \rightarrow \vec{x}' = \mathcal{M} \vec{x}$ for an invertible matrix $\mathcal{M} \in \mathbb{R}^{3 \times 3}$:

This is the most general transformation for the position labels of the K and K' -clocks at $t = 0$. Note that we make no statements on the times t' displayed by the K' -clocks at $t = 0$.

$$\mathcal{M} = R_1 D R_2 = R_1 D R_1^T R = M R \quad (1.41)$$

with $R \in \text{O}(3)$ and $M^T = M$.

This follows from the \downarrow *singular value decomposition* of real matrices with $R_1, R_2 \in \text{O}(3)$ and D a diagonal matrix.

- ii | With spatial rotations Eq. (1.40) we can always transform the K -coordinates by $\vec{x} \mapsto R^{-1} \vec{x}$ such that $\vec{x}' = \mathcal{M} \vec{x} = M \vec{x}$ at $t = 0 \rightarrow$

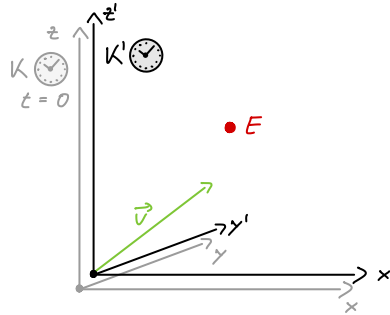
- ** Pure boost** $K \xrightarrow{1, \vec{v}, 0, \vec{0}} K'$:

$$x' = \Lambda_{\vec{v}} x \quad \Leftrightarrow \quad \begin{cases} t' = a(\vec{v}) t + \vec{b}(\vec{v}) \cdot \vec{x} \\ \vec{x}' = M(\vec{v}) \vec{x} + \vec{e}(\vec{v}) t \end{cases} \quad (1.42)$$

- a : \vec{v} -dependent scalar
- \vec{b}, \vec{e} : \vec{v} -dependent vectors
- $M^T = M$: \vec{v} -dependent 3×3 -matrix

Pure boosts are therefore characterized by a symmetric transformation of the spatial coordinates at $t = 0$ in K . Geometrically, this implies that there are *three* (orthogonal) lines through the origin of K which are mapped onto themselves under the boost (spanned by the eigenvectors of $M(\vec{v})$). The only other possibility is that there is a *single* invariant line, which then coincides with the rotation axis of a spatial rotation mixed into the boost. The *pure* boosts are therefore those boosts *without* any rotation mixed in.

→ We focus on pure boosts in the remainder of this derivation:



! Our characterization of a pure boost does *not* imply that at $t = 0$ the axes of the two systems K and K' align (as suggested by the sketch and naïvely expected). If this were the case, the eigenbasis of $M(\vec{v})$ would be given by the basis vectors \hat{e}_i in K . Since we do not know the form of $M(\vec{v})$ (yet), we cannot make this assumption! So do not take this sketch literally, it only illustrates *symbolically* the situation of a pure boost in an arbitrary direction.

6 | Isotropy:

Here are two lines of arguments that use isotropy **IS** to restrict the form of Eq. (1.42) further:

- Argument A:

- i | We claim that isotropy **IS** requires the following multiplicative structure for pure boosts and rotations:

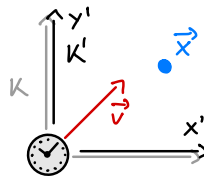
$$\Lambda_R \Lambda_{\vec{v}} \Lambda_{R^{-1}} \stackrel{!}{=} \Lambda_{R\vec{v}} \Leftrightarrow \forall_x : \Lambda_R \Lambda_{\vec{v}} x = \Lambda_{R\vec{v}} x \stackrel{!}{=} \Lambda_{R\vec{v}} \Lambda_R x. \quad (1.43a)$$

$$\Leftrightarrow \forall_x : \Lambda_{\vec{v}} x \stackrel{!}{=} \Lambda_{R^{-1}} \Lambda_{R\vec{v}} (\Lambda_R x). \quad (1.43b)$$

The reasoning goes as follows:

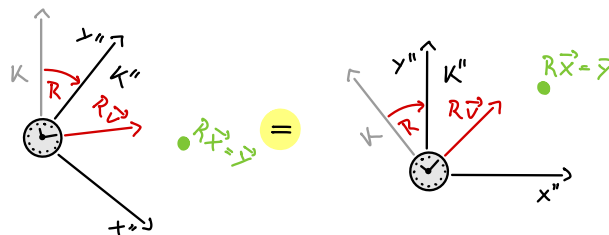
1. \Leftarrow Left-hand side of Eq. (1.43b):

$x = (t, \vec{x})$ are the coordinates of some event in K and $\Lambda_{\vec{v}} x$ of the same event in K' :



2. \Leftarrow Right-hand side of Eq. (1.43b):

We consider $y = (t, \vec{y}) := \Lambda_R x = (t, R\vec{x})$ as an *active* transformation, i.e., y denotes a different event that is spatially rotated from x by R . To state our isotropy claim **IS**, we now rotate the coordinate system K'' in which we want to express this event *in the same way*. This implies a rotated boost $\Lambda_{R\vec{v}}$ and a subsequent rotation of the coordinate axes by R via $\Lambda_{R^{-1}}$. (Remember that when rotating the coordinate axes by R , the *coordinates* of an event transform by $\Lambda_{R^{-1}}$):



3. Spatial isotropy **IS** is the property that the event x as seen from K' cannot be distinguished from the rotated event y as seen from the rotated system K'' ; this is Eq. (1.43b).

ii | Now we can use Eq. (1.42) to rewrite Eq. (1.43a) as

$$t' \stackrel{!}{=} a(R\vec{v})t + \vec{b}(R\vec{v}) \cdot R\vec{x} \quad (1.44a)$$

$$R\vec{x}' \stackrel{!}{=} M(R\vec{v})R\vec{x} + \vec{e}(R\vec{v})t \quad (1.44b)$$

iii | A comparison with Eq. (1.42) (for all t and \vec{x} and arbitrary \vec{v} and R) leads to constraints on the unknown functions:

$$- a(\vec{v}) \stackrel{!}{=} a(R\vec{v}) \rightarrow a(\vec{v}) = a_v \text{ with } v = |\vec{v}|$$

Functions invariant under arbitrary rotations can only depend on the norm $|\vec{v}|$.

$$- \vec{b}(\vec{v}) \stackrel{!}{=} R^T \vec{b}(R\vec{v}) \rightarrow \vec{b}(\vec{v}) = b_v \vec{v}$$

Note that $\vec{b}(R\vec{v}) \cdot R\vec{x} = [R^T \vec{b}(R\vec{v})] \cdot \vec{x}$. Let $R_{\hat{v}}$ be some rotation with axis $\hat{v} = \vec{v}/v$ such that $R_{\hat{v}}\vec{v} = \vec{v}$; then $\vec{b}(\vec{v}) \stackrel{!}{=} R_{\hat{v}}^T \vec{b}(\vec{v})$ and therefore $\vec{b}(\vec{v}) \propto \vec{v}$ since rotation matrices have only a single eigenvector.

$$- RM(\vec{v}) \stackrel{!}{=} M(R\vec{v})R \rightarrow M(\vec{v}) = c_v \mathbb{1} + d_v \hat{v}\hat{v}^T$$

First recall that $M^T(\vec{v}) = M(\vec{v})$ such that $M(\vec{v})$ can be written as sum of orthogonal projectors (projecting onto its eigenspaces). It is in particular $R_{\hat{v}} M(\vec{v}) R_{\hat{v}}^T \stackrel{!}{=} M(\vec{v})$ such that one of the eigenvectors must be $\hat{v} \propto \vec{v}$. The remaining two eigenvectors are orthogonal to \hat{v} and can therefore be mapped onto each other by $R_{\hat{v}}$. Since $R_{\hat{v}}$ commutes with $M(\vec{v})$, their eigenvalues must be degenerate such that the two-dimensional subspace orthogonal to \hat{v} is a degenerate eigenspace. The most general spectral decomposition of $M(\vec{v})$ is then the one given above.

$$- R\vec{e}(\vec{v}) \stackrel{!}{=} \vec{e}(R\vec{v}) \rightarrow \vec{e}(\vec{v}) = e_v \vec{v}$$

This is the same argument as for $\vec{b}(\vec{v})$.

• Argument B:

A shorter (but less rigorous) line of arguments goes as follows:

- i | To define the unknown functions algebraically, we are only allowed to use the vector \vec{v} and constant scalars. We *cannot* use \vec{x} or t due to linearity, and any other constant vector (like $\hat{e}_x = (1, 0, 0)^T$) would pick out some direction and therefore violate isotropy **IS**.
- ii | Since the only *scalar* one can construct from a single vector is its norm, $|\vec{v}|^2 = \vec{v} \cdot \vec{v}$, it must be $a(\vec{v}) = a_v$.
- iii | Similarly, since the only *vector* one can construct from a single vector is a scalar multiplied by the vector itself, it must be $\vec{b}(\vec{v}) = b_v \vec{v}$ and $\vec{e}(\vec{v}) = e_v \vec{v}$.
- iv | Lastly, since $M^T(\vec{v}) = M(\vec{v})$, we can decompose the matrix into orthogonal projectors: $M(\vec{v}) = \sum_i \lambda_i(v) P_i(\vec{v})$. The only projectors that can be defined by a single vector are $P_0 = \hat{v}\hat{v}^T$ and $P_1 = \mathbb{1} - P_0 = \mathbb{1} - \hat{v}\hat{v}^T$ which leads to the most general form $M(\vec{v}) = c_v \mathbb{1} + d_v \hat{v}\hat{v}^T$.

Both arguments lead to the same form for pure boosts $\Lambda_{\vec{v}}$ consistent with isotropy **IS**:

$$t' = a_v t + b_v (\vec{v} \cdot \vec{x}) \quad (1.45a)$$

$$\vec{x}' = c_v \vec{x} + \frac{d_v}{v^2} \vec{v}(\vec{v} \cdot \vec{x}) + e_v \vec{v} t \quad (1.45b)$$

with $v = |\vec{v}| = |R\vec{v}|$ and $(R\vec{v} \cdot R\vec{x}) = (\vec{v} \cdot \vec{x})$.

7 | < Trajectory of origin O' of K' :

- In K' : $\vec{x}'_{O'} = 0$ (This is the operational definition of the origin O' .)
- In K : $\vec{x}_{O'} = \vec{v}t$ (This is the operational definition of \vec{v} in $K \xrightarrow{\mathbb{1}, \vec{v}, 0, \vec{0}} K'$.)

In Eq. (1.45b):

$$\vec{0} = c_v \vec{v}t + \frac{d_v}{v^2} \vec{v}(\vec{v} \cdot \vec{v})t + e_v \vec{v}t \quad (1.46a)$$

$$\vec{v} \neq \vec{0} \ \& \ \forall t \quad \Rightarrow \quad 0 = c_v + d_v + e_v \quad (1.46b)$$

8 | Reciprocity:

- i | < Inverse transformation $K' \xrightarrow{\mathbb{1}, \vec{v}', 0, \vec{0}} K$ from K' to K :

$$\Lambda_{\vec{v}'} \Lambda_{\vec{v}} = \mathbb{1} \quad \Leftrightarrow \quad \Lambda_{\vec{v}'} = \Lambda_{\vec{v}}^{-1}. \quad (1.47)$$

Note that \vec{v}' is the velocity of the origin O of K as measured in K' .

In general: $\vec{v}' = \vec{V}(\vec{v})$ with unknown function \vec{V} .

We assume *reciprocity*: $\vec{v}' = -\vec{v}$ such that

$$\Lambda_{-\vec{v}}^{-1} = \Lambda_{\vec{v}}. \quad (1.48)$$

While this is clearly the most reasonable/intuitive assumption, it is not trivial! Recall that \vec{v} is the speed of the origin O' of K' measured with the clocks in K , whereas \vec{v}' is the speed of the origin O of K measured with *different* clocks in K' . So without additional assumptions we cannot conclude that the results of these measurements yield reciprocal results.

However, the assumption of reciprocity can be rigorously derived from relativity **SR**, isotropy **IS** and homogeneity **HO**, see Ref. [32]. Reciprocity is therefore not an independent assumption.

- ii | < Inverse transformation in Eq. (1.45):

$$t = a_v t' - b_v (\vec{v} \cdot \vec{x}') \quad (1.49a)$$

$$\vec{x} = c_v \vec{x}' + \frac{d_v}{v^2} \vec{v}(\vec{v} \cdot \vec{x}') - e_v \vec{v}t' \quad (1.49b)$$

- iii | Eq. (1.49) in Eq. (1.45) & Eq. (1.46b) $\overset{\circ}{\Rightarrow}$ (we suppress the v dependence)

$$c^2 = 1, \quad (1.50a)$$

$$a^2 - ebv^2 = 1, \quad (1.50b)$$

$$e^2 - ebv^2 = 1, \quad (1.50c)$$

$$e(a + e) = 0, \quad (1.50d)$$

$$b(a + e) = 0. \quad (1.50e)$$

To show this, use $\vec{v} = (v_x, 0, 0)^T$ with $v_x \neq 0$ and remember that the equations you obtain from plugging Eq. (1.49) into Eq. (1.45) must be valid for all t' and \vec{x}' . Use Eq. (1.46b) to replace $c_v + d_v$ by $-e_v$.

We can conclude:

- $\xrightarrow{\text{Eq. (1.50a)}} c = 1$ ($c = -1$ contradicts $\lim_{v \rightarrow 0} \Lambda_{\vec{v}} \stackrel{!}{=} \mathbb{1}$)
- $\xrightarrow{\text{Eq. (1.50c)}} e \neq 0 \xrightarrow{\text{Eq. (1.50d)}} a + e = 0$
→ Eq. (1.50b) \equiv Eq. (1.50c) & Eq. (1.50e) satisfied

9 | Collecting results from Eq. (1.50) & Eq. (1.46b):

$$c = 1, \quad e = -a, \quad d = a - 1, \quad b = \frac{1-a^2}{av^2}. \quad (1.51)$$

$d = a - 1$ follows from Eq. (1.46b) and the first two equations.

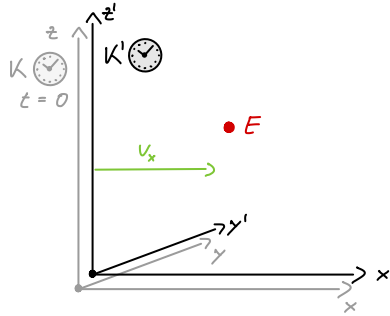
$$\text{Eq. (1.45)} \xrightarrow{\text{Eq. (1.51)}}$$

$$t' = a_v t + \frac{1-a_v^2}{va_v} (\hat{v} \cdot \vec{x}) \quad (1.52a)$$

$$\vec{x}' = \vec{x} + [a_v - 1] \hat{v} (\hat{v} \cdot \vec{x}) - va_v \hat{v} t \quad (1.52b)$$

with $\hat{v} := \vec{v}/|\vec{v}|$.

10 | < Special boost $\vec{v} = (v_x, 0, 0)^T$ in x -direction:



$$t' = a_v t + \frac{1-a_v^2}{v_x a_v} x \quad (1.53a)$$

$$x' = a_v x - v_x a_v t \quad (1.53b)$$

$$y' = y \quad (1.53c)$$

$$z' = z \quad (1.53d)$$

Note that $v = |v_x|$ with $v_x \in \mathbb{R}$.

Matrix form:

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \underbrace{\begin{pmatrix} a_v & \frac{1-a_v^2}{v_x a_v} & & \\ -v_x a_v & a_v & & \\ & & 1 & 0 \\ & & 0 & 1 \end{pmatrix}}_{=:\Lambda_{v_x}} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \quad (1.54)$$

In the following, we refer to the upper 2×2 -block as $A(v_x)$.

11 | Group structure:

i | Relativity principle **SR** →

$$\varphi(K' \xrightarrow{R_2, \vec{v}_2, s_2, \vec{b}_2} K'') \circ \varphi(K \xrightarrow{R_1, \vec{v}_1, s_1, \vec{b}_1} K') \stackrel{!}{=} \varphi(K \xrightarrow{R_3, \vec{v}_3, s_3, \vec{b}_3} K'') \quad (1.55)$$

for some parameters $(R_3, \vec{v}_3, s_3, \vec{b}_3)$ that are a function of $(R_i, \vec{v}_i, s_i, \vec{b}_i)_{i=1,2}$.

In words:

The concatenation of a coordinate transformations from K to K' and from K' to K'' must be another coordinate transformation *that is parametrized by data that relates the reference systems K with K'' directly (without referring to K' in any way).*

You may ask why Eq. (1.55) is a constraint on φ in the first place. After all, we could just define that

$$\varphi(K \xrightarrow{R_3, \vec{v}_3, s_3, \vec{b}_3} K'') := \varphi(K' \xrightarrow{R_2, \vec{v}_2, s_2, \vec{b}_2} K'') \circ \varphi(K \xrightarrow{R_1, \vec{v}_1, s_1, \vec{b}_1} K'). \quad (1.56)$$

The problem is that the function defined such generically depends on 8 (!) parameters $R_1, \vec{v}_1, s_1, \vec{b}_1, R_2, \vec{v}_2, s_2, \vec{b}_2$ – it is a non-trivial functional constraint on φ that these can be compressed to four parameters $R_3, \vec{v}_3, s_3, \vec{b}_3$. This “compression” is mandated by the relativity principle **SR** according to which all inertial systems must be treated equally. In particular, the transformation between two systems K and K'' can only depend on parameters that can be experimentally determined from within these two systems. (The existence of) a third frame K' cannot be of relevance for this transformation as this would make K' special.

Combined with the existence of an inverse transformation (\leftarrow above):

→ The set of all transformations forms a \downarrow (multiplicative) group.

Note that *associativity* is implicit since we talk about the concatenation of linear/affine maps.

ii | In particular:

$$\Lambda_{v_x} \Lambda_{u_x} \stackrel{!}{=} \Lambda_{w_x} \quad \Leftrightarrow \quad A(v_x)A(u_x) \stackrel{!}{=} A(w_x) \quad (1.57)$$

where $w_x = W(v_x, u_x)$ has to be determined.

- \uparrow ! Using the restricted form of the boost Eq. (1.54) that followed from previous arguments, it follows indeed that the concatenation of two pure boosts *in the same direction* has again the form of a pure boost (in the same direction). For the arguments that follow, this is sufficient.

However, in general, the multiplicative group structure Eq. (1.55) allows for two boosts to concatenate to a *combination* of boosts and rotations. As we will see \rightarrow later, this is indeed what happens: The concatenation of two pure boosts (in different directions) produces a boost with a rotation mixed in (\uparrow *Thomas-Wigner rotation*).

- Note that due to Eq. (1.43a) all that follows holds for any pair of *collinear* velocities \vec{v} and \vec{u} (there is nothing special about the x -direction). Indeed, let R be a rotation that maps \vec{v} and \vec{u} to vectors on the x -axis, $\vec{v}_x := R\vec{v}$ and $\vec{u}_x := R\vec{u}$. Then

$$\Lambda_{\vec{v}} \Lambda_{\vec{u}} \stackrel{1.43a}{=} \Lambda_{R^{-1}} \Lambda_{\vec{v}_x} \Lambda_{\vec{u}_x} \Lambda_R \stackrel{!}{=} \Lambda_{R^{-1}} \Lambda_{\vec{w}_x} \Lambda_R \stackrel{1.43a}{=} \Lambda_{\vec{w}} \quad (1.58)$$

where \vec{w} is again collinear with \vec{v} and \vec{u} .

$\xrightarrow{\circ}$ (use that the diagonal elements of $A(w_x)$ must be equal)

$$\forall_{v_x, u_x} : \quad \frac{1 - a_v^2}{v_x^2 a_v^2} \stackrel{!}{=} \frac{1 - a_u^2}{u_x^2 a_u^2} \quad (1.59)$$

→ Universal constant:

$$\kappa := \frac{a_v^2 - 1}{v_x^2 a_v^2} = \text{const} \quad (1.60)$$

Note: $[\kappa] = \text{Velocity}^{-2}$

→

$$a_v = \frac{1}{\sqrt{1 - \kappa v_x^2}}. \quad (1.61)$$

We use the positive solution for a_v since $\lim_{v \rightarrow 0} A(v) \stackrel{!}{=} 1$, i.e., $\lim_{v \rightarrow 0} a_v \stackrel{!}{=} 1$.

iii | With this we check: $A(v_x)A(u_x) \stackrel{!}{=} A(w_x)$ with

$$w_x = W(v_x, u_x) \stackrel{!}{=} \frac{v_x + u_x}{1 + u_x v_x \kappa}. \quad (1.62)$$

Eq. (1.62) becomes important later: it tells us how to add velocities in **SPECIAL RELATIVITY**.

12 | Preliminary result:

Eq. (1.52) & Eq. (1.60) → Boost $\Lambda_{\vec{v}}$ in direction \hat{v} with velocity $\vec{v} = v \hat{v}$:

$$t' = a_v [t - \kappa (\vec{v} \cdot \vec{x})] \quad (1.63a)$$

$$\vec{x}' = \vec{x} + [a_v - 1] \hat{v} (\hat{v} \cdot \vec{x}) - a_v \vec{v} t \quad (1.63b)$$

with

$$a_v = \frac{1}{\sqrt{1 - \kappa v^2}}. \quad (1.64)$$

This is the most general transformation between two inertial coordinate systems that move with relative velocity \vec{v} (with coinciding axes at $t = 0$) that is consistent with our basic assumptions stated at the beginning of this section: **SR**, **HO**, and **IS**.

The only undetermined parameter left is κ .

1.5. The Lorentz transformation

The purpose of this section is to select the value for κ that describes our reality.

13 | Since $[\kappa] = \text{Velocity}^{-2}$ define formally: $\kappa \equiv 1/v_{\max}^2$.

Why we subscribe the velocity v_{\max} with “max” will become clear below.

14 | Three cases:

- $\kappa = 0 \Leftrightarrow v_{\max} = \infty$:

$$\text{Eq. (1.63)} \Rightarrow \left. \begin{array}{l} t' = t \\ \vec{x}' = \vec{x} - \vec{v} t \end{array} \right\} \quad \text{** Galilei boost} \quad (1.65a)$$

→ Maxwell equations are *not* form-invariant under φ .

- Maxwell equations cannot be correct and must be modified.
- Experiment that shows the invalidity of Maxwell equations?

Note that we cannot conclude the *validity* of classical mechanics from this; Newton's equations may still require modifications (without spoiling the Galilean symmetry, of course).

- $\kappa > 0 \Leftrightarrow v_{\max} < \infty$:

$$\text{Eq. (1.63)} \Rightarrow \left. \begin{aligned} t' &= \gamma \left(t - \frac{\vec{v} \cdot \vec{x}}{v_{\max}^2} \right) \\ \vec{x}' &= \vec{x} + (\gamma - 1) \hat{v}(\hat{v} \cdot \vec{x}) - \gamma \vec{v} t \end{aligned} \right\} \text{** Lorentz boost}$$

(1.66a)

with the ** Lorentz factor

$$\gamma_v \equiv \gamma := \frac{1}{\sqrt{1 - \beta^2}} \quad \text{and} \quad \beta := v/v_{\max} . \quad (1.67)$$

- Newton's equations are *not* form-invariant under φ .
- Classical mechanics cannot be correct and must be modified.
- Experiment that shows the invalidity of Newton's equations?

Similarly, we cannot conclude the *validity* of electrodynamics from this; Maxwell equations may still require modifications (without spoiling the Lorentz symmetry).

- $\kappa < 0$: Physically not relevant. (⊖ Problemset 2; we ignore this solution in the following.)

This solution is not self-consistent (see e.g. Ref. [31]) and immediately leads to implications that are not observed in nature.

For example, the rule Eq. (1.62) to compute the velocity w_x between K/K'' from the velocities v_x and u_x between K/K' and K'/K'' reads for $\kappa < 0$

$$w_x = \frac{v_x + u_x}{1 - u_x v_x |\kappa|} . \quad (1.68)$$

Let $u_x, v_x > 0$ be positive, i.e., K' moves in *positive* x -direction wrt K and K'' moves also in *positive* x -direction wrt K' . But for large enough velocities $u_x v_x > 1/|\kappa|$ we find $w_x < 0$ such that K'' moves in *negative* x -direction wrt K .

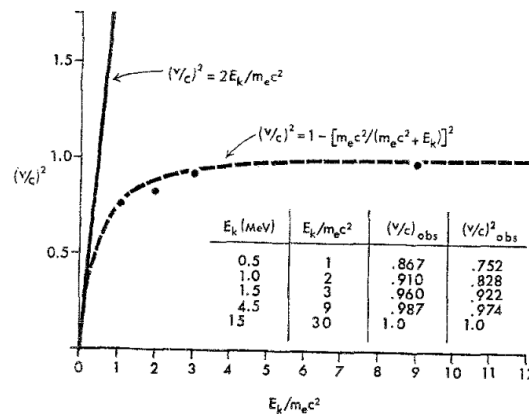
No such effect has ever been observed; if you do, let us know!

Note that at no point we used or claimed that v_{\max} is the speed of light!

Which transformation describes reality: $v_{\max} < \infty$ or $v_{\max} = \infty$?

15 | Evidence:

- Maximum velocity $v_{\max} \approx c < \infty$ for electrons (plot from Ref. [33]):



→ Newton’s equations are clearly invalid for high velocities!

See Refs. [33, 34] for more technical details. Note that these results were obtained decades after Einstein published his seminal paper in 1905.

• *By contrast:*

No evidence for the invalidity of Maxwell equations (on the macroscopic level).

Electrodynamics, as encoded by the Maxwell equations, is of course not a truly fundamental theory as it is the classical limit of a quantum theory: Quantum electrodynamics (QED). For example, the linearity of the Maxwell equations (= EM waves cannot scatter off each other) is an *approximation*; in QED photons *can* (weakly) scatter off each other! This is why I emphasize that Maxwell theory is experimentally valid only on the *macroscopic level*. Note, however, that QED has the same spacetime symmetry group as electrodynamics, namely Lorentz transformations.

16 | Hence it is reasonable stipulate $v_{\text{max}} < \infty$ and postulate:

The transformations φ between inertial systems are given by *Lorentz transformations*.
These transformations must be (part of) the spacetime symmetries of *all* physical theories.

The last statement is often rephrased as follows:

All (fundamental) theories must be *form-invariant* (covariant) under Lorentz transformations.

This is just **SR** all over again: The equations of models that describe reality must “look the same” (more precisely: be functionally equivalent) in all inertial systems. Since the transformations between inertial systems are given by Lorentz transformations (and not Galilean transformations, as historically anticipated), this requires their form-invariance under Lorentz transformations.

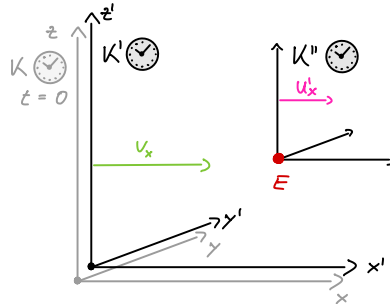
→ **SPECIAL RELATIVITY** restricts the structure of *all fundamental theories of physics!*

This is what is meant by the statement that **SPECIAL RELATIVITY** is a *theoretical framework* (German: *Rahmentheorie*) or “meta theory”: It provides a “recipe” (*ordering principle*) of how to construct consistent theories of physics. The Standard Model of particle physics, for example, is form-invariant under Lorentz transformations, and if you propose an extension thereof (for example to give neutrinos a mass) you better make sure that the terms you write down are also form-invariant under Lorentz transformations (otherwise you will not be taken seriously!). Note,

however, that this perspective prevents an important insight: What we really study is an entity called *spacetime*, and this entity has a property: Lorentz symmetry. Since all our (fundamental) physical theories are formulated *on* spacetime, it should not come as a surprise that the Lorentz symmetry of spacetime shows up all over the place.

17 | Interpretation of v_{\max} :

- i | \leftarrow Systems $K \xrightarrow{v_x} K'$ and signal with velocity $\frac{dx'}{dt'} = u'_x$:



Question: What is the velocity $u_x = \frac{dx}{dt}$ of this signal in K ?

- ii | Remember (Group structure!):

$$\varphi(K' \xrightarrow{v_2} K'') \circ \varphi(K \xrightarrow{v_1} K') = \varphi(K \xrightarrow{v_3} K'') \quad \text{with} \quad v_3 = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{v_{\max}^2}}. \quad (1.69)$$

Let $v_1 = v_x$ and $v_2 = u'_x$ so that $v_3 = u_x$ (i.e., the signal is at rest in the origin of K'').

You can also derive this by computing the time derivative of the position of the signal in K using a Lorentz transformation; you will do this properly when you derive a more general addition of velocities (→ Problemset 2).

- iii | Addition formula for collinear velocities:

$$u_x = \frac{v_x + u'_x}{1 + \frac{v_x u'_x}{v_{\max}^2}} \quad (1.70)$$

Because of isotropy **IS** this formula must be true in all directions (not just in x -direction) as long as the two velocities to be added are parallel. We still keep the index x to signify that these are not absolute values of velocities.

- Note that for $v_{\max} \rightarrow \infty$ we get back the “conventional” (= Galilean) additivity of velocities:

$$u_x = (v_x + u'_x) \left[1 - \frac{v_x u'_x}{v_{\max}^2} + \dots \right] \stackrel{v_{\max} \rightarrow \infty}{\approx} v_x + u'_x \quad (1.71)$$

From this expansion and the validity of classical mechanics for small velocities (in particular its law for adding velocities), we can also conclude that v_{\max} must be *large compared to everyday experience*.

- A historically influential experiment that (in hindsight) can be explained by the relativistic addition of velocities Eq. (1.70) is the \uparrow *Fizeau experiment* [35, 36] (see also \uparrow *Fresnel drag coefficient*). The Fizeau experiment was one of the crucial hints that led Einstein to SPECIAL RELATIVITY.

iv | $0 \leq v_x, u'_x \leq v_{\max}$: ($\tilde{v}_x := v_x/v_{\max}$ so that $0 \leq \tilde{v}_x, \tilde{u}'_x \leq 1$)

$$u_x = v_{\max} \frac{\tilde{v}_x + \tilde{u}'_x}{1 + \tilde{v}_x \tilde{u}'_x} \leq v_{\max} \quad (1.72)$$

Here we used that $a + b \leq 1 + ab$ for numbers $0 \leq a, b \leq 1$.

→ “Addition” of velocities Eq. (1.70) never exceeds v_{\max} .

→ v_{\max} plays the role of a *maximum velocity*.

v | \triangleleft Signal with maximum velocity in K' : $u'_x = v_{\max}$:

$$u_x = \frac{v_{\max} + v_x}{1 + \frac{v_{\max} v_x}{v_{\max}^2}} = v_{\max} \frac{v_{\max} + v_x}{v_{\max} + v_x} = v_{\max} \quad (1.73)$$

Note that the result is completely independent of the velocity v_x of K' !

→ Whatever moves with the maximum velocity v_{\max} does so *in all inertial systems*!

Please appreciate how counterintuitive this effect is *from the perspective of everyday experience*!

But also notice that we didn't have to postulate it: The relativity principle **SR** together with the *existence* of a (finite) maximum velocity is sufficient.

If you think about it: Assuming a maximum velocity (in the absence of a preferred reference frame) automatically invalidates the simple Galilean law of additive velocities. So it is actually not surprising at all that the maximum velocity must be independent of the reference system.