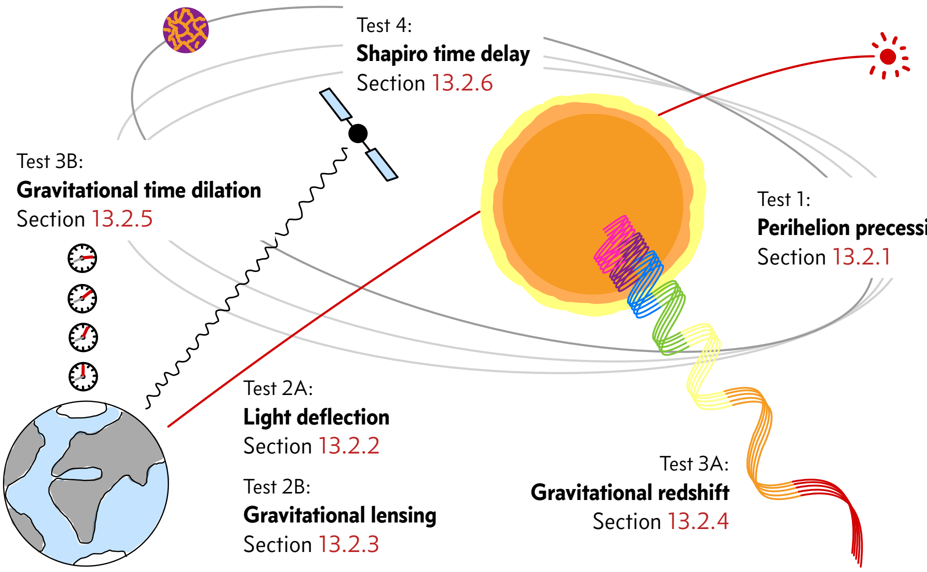
↓Lecture 28 [09.07.24]

13.2. Tests of GENERAL RELATIVITY in the Solar System

With the Schwarzschild metric at hand, we can finally derive predictions of GENERAL RELATIVITY that can be used to distinguish the theory from its non-relativistic predecessor, Newtonian dynamics. Here we focus on tests and predictions that are applicable to scales within our Solar System. Hence we omit the cosmological constant ($\Lambda = 0$) and can also safely assume $r \gg r_s$, so that the singularities of the Schwarzschild metric can be ignored:



Einstein introduced and studied 1916 in Ref. [20] (§22, pp. 818–822) what are today known as the *"Three classical tests of* GENERAL RELATIVITY*"*. He summarized and popularized them 1919 in an article written for the London Times [213, 214]:

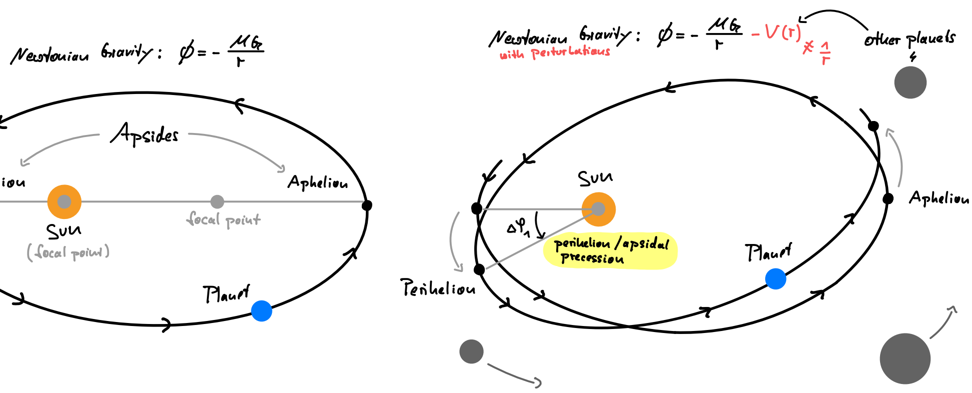
- The perihelion precession of Mercury (→ Section 13.2.1)
- The deflection of light by the Sun (→ Section 13.2.2)
- The gravitational redshift of light (→ Section 13.2.4)

In Einstein's words [213] (p. 209):

> *Die neue Theorie der Gravitation weicht in prinzipieller Hinsicht von der Theorie Newtons bedeutend ab. Aber ihre praktischen Ergebnisse stimmen mit denen der Newton'schen Theorie so nahe überein, dass es schwer fällt, Unterscheidungs-Kriterien zu finden, die der Erfahrung zugänglich sind. Solche haben sich bis jetzt gefunden*
>
> *1) in der Drehung der Ellipsen der Planetenbahnen um die Sonne (beim Merkur bestätigt).*
>
> *2) in der Krümmung der Lichtstrahlen durch die Gravitationsfelder (durch die englischen Sonnenfinsternis-Aufnahmen bestätigt).*
>
> *3) in einer Verschiebung der Spektrallinien nach dem roten Spektralende hin des von Sternen bedeutender Masse zu uns gesandten Lichtes (bisher nicht bestätigt).*
>
> *Der Hauptreiz der Theorie liegt in ihrer logischen Geschlossenheit. Wenn eine einzige aus ihr gezogene Konsequenz sich als unzutreffend erweist, muss sie verlassen werden; eine blosse Modifikation erscheint ohne Zerstörung des ganzen Gebäudes unmöglich.*

13.2.1. Apsidal precession

The first and most famous application of GENERAL RELATIVITY was and is the explanation of the anomalous apsidal precession of Mercury's orbit:



Problem:

Taking into account all known gravitational perturbations (mostly due to other planets) explains Mercury's apsidal precession up to a deviation of [215]

$$\Delta\varphi_? \approx (42.56 \pm 0.94)'' \text{ per century} \tag{13.44}$$

which remains mysterious in Newton's theory ☹.

Solution: GENERAL RELATIVITY ☺

The fact that GENERAL RELATIVITY can be used to compute $\Delta\varphi_?$ precisely was a triumph for Einstein, and paved the way for a quick adoption of the theory. Einstein derived $\Delta\varphi_?$ in his famous paper *"Erklärung der Perihelbewegung des Merkur aus der allgemeinen Relativitätstheorie"* [13], published on 18. November 1915.

If you look sharply at the publication date, you might wonder how Einstein was able to pull off this feat if his foundational paper *"Die Feldgleichungen der Gravitation"* [12] (in which he published the Einstein field equations) appeared *later*, namely on 25. November 1915. The reason is that he used the "wrong" equations $R_{\mu\nu} = -\kappa T_{\mu\nu}$ [which he introduced in Ref. [11] on 11. November 1915 (with a correction added on 18. November), see Eq. (16b) on p. 800 (remember that Einstein's notation differs from ours)] to do the Mercury calculation. Because this calculation rests on the *vacuum* field equations only, and $G_{\mu\nu} = 0 \Leftrightarrow R_{\mu\nu} = 0$, these results remained unaffected by his later modification of the field equations. Einstein writes in Ref. [12]:

> *Die Feldgleichungen für das Vakuum, auf welche ich die Erklärung der Perihelbewegung des Merkur gegründet habe, bleiben von dieser Modifikation* [the addition of the term $\frac{1}{2}g_{\mu\nu}T$ in the trace-inverted form Eq. (12.11)] *unberührt.*

‡ Reminder: The Kepler problem in Newtonian mechanics

Let us first revisit the two-body problem in Newtonian mechanics so that we can compare it to the modifications due to the Schwarzschild geometry later:

1 | System: ⊲ Test mass m in gravitational field of heavy mass $M \gg m$:

We use spherical coordinates (r, θ, φ) on Euclidean space to exploit the rotational symmetry.

Rotational symmetry → Conservation of angular momentum → *w.l.o.g.* $\theta = \frac{\pi}{2}$

→ Lagrangian of test mass:

$$L = \underbrace{\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\varphi}^2\right)}_{\text{Kinetic energy}} + \underbrace{\frac{GmM}{r}}_{\text{Gravitational energy}} \tag{13.45}$$

2 | Integration:

Integrating the equations of motion of this system is simplified by exploiting its symmetries:

i | φ cyclic (the Lagrangian does not depend on φ) →

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{\varphi}} = 0 \quad\Rightarrow\quad l := m\underbrace{r^2\dot{\varphi}}_{=:h} = \text{const} \tag{13.46}$$

→ Angular momentum l is conserved

ii | Eq. (13.45) translation-symmetric in time t →

$$E := H = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - \frac{GmM}{r} = \text{const} \tag{13.47}$$

→ Energy E is conserved

iii | Use $h = r^2\dot{\varphi} = \text{const}$ and assume $r = r(\varphi) \to \dot{r} = \frac{\mathrm{d}r}{\mathrm{d}\varphi}\dot{\varphi}$

The assumption $r = r(\varphi)$ restricts the set of solutions to the ones we are interested in. There are of course also radial solutions with $\varphi = \text{const}$ and $r = r(t)$, but these are not important for our application to describe planets in the Solar System.

$$\text{Eq. (13.47)} \to \quad E = \frac{1}{2}m\left[\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2\frac{h^2}{r^4} + \frac{h^2}{r^2}\right] - \frac{GmM}{r} \tag{13.48}$$

iv | $\sphericalangle$ New radial coordinate $u := \frac{1}{r} \to u' := \frac{\mathrm{d}u}{\mathrm{d}\varphi} = -\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\varphi}$

$$\text{Eq. (13.48)} \to \quad E = \frac{1}{2}mh^2\left[\left(u'\right)^2 + u^2\right] - GmMu \tag{13.49}$$

v | Assume $u' \neq 0$ and derive Eq. (13.49) wrt. φ:

Solutions with $u' = 0$ imply $r = \text{const}$ and correspond to circular orbits.

$$\text{Eq. (13.49)} \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}\varphi}} \quad u'' + u \stackrel{\circ}{=} A \quad \text{with} \quad A := \frac{GM}{h^2} \tag{13.50}$$

We will find a similar (but modified) equation of this form in the Schwarzschild geometry.

3 | Solution:

Adding homogeneous solutions to the particular solution A of Eq. (13.50) yields the general solution:

$$u = \frac{1}{r} = A\left[1 + e\cos(\varphi - \varphi_0)\right] \tag{13.51}$$

→ ↓ *Conic sections*

For $0 < e < 1$ the orbit $r = r(\varphi)$ describes an *ellipse* with perihelion at $\varphi = \varphi_0$ (*w.l.o.g.* $\varphi_0 = 0$) and *eccentricity* e. For $e = 0$ one obtains the circular solution with radius A^{-1}.

That Eq. (13.51) describes ellipses for $0 < e < 1$ with eccentricity e is not obvious because this equation is the ↓ *polar form* of the ellipse equation with φ measured wrt. one of the ↓ *foci* of the ellipse. (Remember that we put the heavy mass in the origin $r = 0$ of our coordinate system.)

The Kepler problem in Schwarzschild spacetime

We can now tackle the same problem (that is, the motion of a test mass in the gravitational field of a much heavier body) in GENERAL RELATIVITY by using …

- … the Schwarzschild metric produced by the Sun (Section 13.1.3).
- … that the test mass follows geodesics in this metric (Section 11.2).

4 | System:

The geodesic equation follows from the Lagrangian: [← Eq. (10.126) ff. in Section 10.3.3]

$$L = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \quad \text{with Schwarzschild metric } g_{\mu\nu} \tag{13.52}$$

$\lessdot$ Schwarzschild coordinates $(x^0, x^1, x^2, x^3) = (ct, r, \theta, \varphi) \xrightarrow{\text{Eq. (13.25)}}$

$$L = \frac{1}{2}\left[\left(1-\frac{r_s}{r}\right)c^2\dot{t}^2 - \left(1-\frac{r_s}{r}\right)^{-1}\dot{r}^2 - r^2\left(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta\right)\right] \tag{13.53}$$

with $\dot{\square} := \frac{\mathrm{d}\square}{\mathrm{d}\tau}$ and proper time τ.

- We can parametrize the geodesic with proper time because $m \neq 0$ for the test mass.
- We could also plug the Christoffel symbols Eq. (13.10) [together with Eq. (13.20)] into the geodesic equation Eq. (10.131) and solve it. Here we follow a more pedestrian (and less technical) approach to work out the differences to the Newtonian case above.

5 | Equation of motion:

i | $\lessdot$ Euler-Lagrange equation for $x^2 = \theta$:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \quad \overset{\circ}{\Leftrightarrow} \quad \ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \dot{\varphi}^2 \sin\theta\cos\theta = 0 \tag{13.54}$$

Solved by $\theta = \frac{\pi}{2} = \text{const}$ (without imposing restrictions on the other coordinates!)

As before, this restricts our solutions to the plane with $\theta = \frac{\pi}{2}$; because of the spherical symmetry of the problem this no actual restriction.

ii | This choice simplifies the Lagrangian:

$$\text{Eq. (13.53)} \xrightarrow{\theta=\frac{\pi}{2}} \quad L_1 = \frac{1}{2}\left[\left(1-\frac{r_s}{r}\right)c^2\dot{t}^2 - \left(1-\frac{r_s}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\varphi}^2\right] \tag{13.55}$$

The subscript reminds us that this Lagrangian only describes motions in the $\theta = \frac{\pi}{2}$ plane.

iii | $\lessdot$ Euler-Lagrange equation for $x^1 = r \overset{\circ}{\rightarrow}$

$$\left(1-\frac{r_s}{r}\right)^{-1}\ddot{r} + \frac{1}{2}\frac{r_s}{r^2}c^2\dot{t}^2 - \frac{1}{2}\left(1-\frac{r_s}{r}\right)^{-2}\frac{r_s}{r^2}\dot{r}^2 - r\dot{\varphi}^2 = 0 \tag{13.56}$$

This complicated EOM is not needed because we exploit enough integrals of motion (→ *below*).

iv | Cyclic coordinates $x^0 = ct$ and $x^3 = \varphi$ → Integrals of motion:

$$\frac{\partial L_1}{\partial(c\dot{t})} = \left(1-\frac{r_s}{r}\right)c\dot{t} =: k = \text{const} \quad \text{and} \quad \frac{\partial L_1}{\partial \dot{\varphi}} = r^2\dot{\varphi} =: h = \text{const} \tag{13.57}$$

v | $m \neq 0 \xrightarrow{\text{Eq. (11.51)}} \|\dot{x}\|^2 = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = c^2 > 0$

With this we restrict our derivation to *time-like* solutions (as needed for a massive test particle). The fact that $\|\dot{x}\|^2 = \text{const}$ was proven in Eq. (11.3) and is a consequence of $x^\mu(\tau)$ describing a geodesic *and* τ being an ← *affine parameter* [which is true for all solutions of the geodesic equation Eq. (10.131)]. That the constant equals c^2 selects a specific affine parameter, namely the *proper time* τ.

$$g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = c^2 \xRightarrow[\theta=\frac{\pi}{2}]{13.25} \left(1-\frac{r_s}{r}\right)c^2\dot{t}^2 - \left(1-\frac{r_s}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\varphi}^2 = c^2 \tag{13.58}$$

vi | Eqs. (13.57) and (13.58) $\overset{\circ}{\rightarrow}$

$$\frac{1}{r^4}\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 + \frac{1}{r^2}\left(1-\frac{r_s}{r}\right)\left(1+\frac{c^2r^2}{h^2}\right) - \frac{k^2}{h^2} = 0 \tag{13.59}$$

Here we assumed $\dot{\varphi} \neq 0$ (thereby excluding radial motions) and used the chain rule,

$$\frac{\dot{r}^2}{\dot{\varphi}^2} = \left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 , \tag{13.60}$$

once again imposing a restriction to solutions of the form $r = r(\varphi)$.

vii | $\triangleleft$ New radial coordinate $u := \frac{1}{r}$

$$\text{Eq. (13.59)} \overset{\circ}{\rightarrow} \quad (u')^2 + u^2 = \frac{k^2 - c^2}{h^2} + \frac{c^2 r_s}{h^2}u + r_s u^3 \tag{13.61}$$

Here we used again $u' := \frac{\mathrm{d}u}{\mathrm{d}\varphi} = -\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\varphi}$.

viii | Assume again $u' \neq 0$ and derive Eq. (13.61) wrt. φ:

Solutions with $u' = 0$ imply $r = \text{const}$ and correspond to circular orbits.

$$\text{Eq. (13.61)} \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}\varphi}} \quad u'' + u = A \underbrace{+ \frac{3}{2}r_s u^2}_{\text{cf. (13.50)}} \tag{13.62}$$

with $A = \frac{GM}{h^2}$ and $r_s = \frac{2GM}{c^2}$.

Note that the new term couples with the length scale r_s of GENERAL RELATIVITY and makes the differential equation *non-linear*.

Both the Newtonian source A and the relativistic correction $\propto r_s$ on the right-hand side include Newtons gravitational constant G. However, only r_s contains the speed of light c, which marks this correction as *relativistic*.

Perihelion precession in GENERAL RELATIVITY

6 | Approximate solution:

Up to this point our derivation is exact. However, the differential equation Eq. (13.62) is no longer linear and hard to solve exactly. Thus we apply some (well justified) approximations to solve it:

i | Note that Eq. (13.62) can be written as

$$u'' + \left(1 - \frac{3}{2}\frac{r_s}{r}\right) u = A \tag{13.63}$$

so that the deviations of the Newtonian case Eq. (13.50) are controlled by $\frac{r_s}{r}$.

In the Solar System this ratio is very small:

$$\left(\frac{r_s}{r}\right)_{\text{Mercury}} \sim 7 \times 10^{-8} \tag{13.64}$$

Here we used the Schwarzschild radius $r_s = 3 \times 10^3$ m of the Sun and the perihelion $r \sim 4.6 \times 10^{10}$ m of Mercury.

→ $\sphericalangle\, r_s u^2$ a *perturbation* for the Newtonian solution (13.51),

$$u_0 := A\left[1 + e\cos\varphi\right]. \tag{13.65}$$

→ First-order perturbation:

$$u'' + u \approx A + \tfrac{3}{2} r_s A^2 \underbrace{\left[1 + 2e\cos\varphi + e^2\cos^2\varphi\right]}_{u_0^2} \tag{13.66}$$

Here we inserted the unperturbed solution u_0 into the perturbation of the EOM. Solving this equation yields a first-order correction. One could then reinsert this solution into the equation and repeat the procedure until one converges to a fixed point – and thereby a solution of the non-linear equation. For our purpose, the first-order correction is already sufficient.

ii | The eccentricity of most planets is very small (= their orbits are almost circular). For example: $e_{\text{Mercury}} \approx 0.2 \to$ drop $\mathcal{O}(e^2)$ terms →

$$u'' + u - A \approx 3 r_s A^2 e \cos\varphi \tag{13.67}$$

Here we also dropped the constant $\frac{3}{2} r_s A^2$ on the right-hand side because it can be absorbed into a (small) shift of the constant A on the left-hand side (which will not affect our conclusions below).

iii | Eq. (13.67) is linear → Use unperturbed solution u_0 for ansatz:

$$u \approx u_0 + u_1 \quad \Rightarrow \quad \underbrace{\left(u_0'' + u_0 - A\right)}_{\overset{13.50}{=} 0} + u_1'' + u_1 = 3 r_s A^2 e \cos\varphi \tag{13.68}$$

$\overset{\circ}{\to}$ Particular solution:

$$u_1 = \tfrac{3}{2} r_s A^2 e\, \varphi\, \sin\varphi \tag{13.69}$$

→ Solution:

$$u \approx u_0 + u_1 \overset{13.65}{=} A\Big[1 + e\cos\varphi + \underbrace{\tfrac{3}{2} r_s A\varphi}_{=:\Delta(\varphi)}\, e \sin\varphi\Big] \tag{13.70}$$

iv | The prefactor of $e\sin\varphi$ is very small:

$$\Delta(\varphi) = \tfrac{3}{2} r_s A\varphi = 3\left(\frac{GM}{ch}\right)^2 \varphi \ll 1 \tag{13.71}$$

Recall that $h = r^2\dot{\varphi}$ [← Eq. (13.57)]. Since the motion of planets is non-relativistic, we can estimate $h \sim (1\,\mathrm{AU})^2\,(2\pi/1\,\mathrm{yr}) \sim 10^{15}\,\mathrm{m}^2/\mathrm{s}$. Plugging in the other constants and the mass of the Sun yields $\frac{3}{2}r_s A \sim 10^{-7}$, which, together with $\varphi \in [0, 2\pi) \sim 1$, justifies the following approximation:

$$\text{Eq. (13.70)} \overset{\circ}{\to} \quad u \approx A\,\{1 + e\cos[\varphi - \Delta(\varphi)]\} \tag{13.72a}$$

$$= A\Big\{1 + e\cos\big[\underbrace{\left(1 - \tfrac{3}{2} r_s A\right)}_{<1}\varphi\big]\Big\} \tag{13.72b}$$

Here we used $\cos\Delta(\varphi) \approx 1$ and $\sin\Delta(\varphi) \approx \Delta(\varphi)$, together with the trigonometric identity $\cos[\varphi - \Delta(\varphi)] = \cos\varphi\cos\Delta(\varphi) + \sin\varphi\sin\Delta(\varphi)$.

→ Rosetta orbit (← *Sketch above*)

To see that this equation describes a Rosetta orbit note that $\Delta(\varphi)$ plays the role of an *angle-dependent* phase shift. Hence the object follows ellipses that slowly rotate themselves with φ about the focus point in which the central mass resides.

7 | Perihelion precession:

To evolve from one perihelion to the next, the argument of the cosine in Eq. (13.72b) must advance by 2π (because then the radial distance $u = \frac{1}{r}$ is again the same). →

$$\left(1 - \tfrac{3}{2} r_s A\right)\varphi_1 \equiv 2\pi \quad \Leftrightarrow \quad \varphi_1 = \frac{2\pi}{1 - \frac{3}{2} r_s A} \overset{\text{Taylor}}{\approx} 2\pi + \underbrace{3\pi r_s A}_{=:\Delta\varphi_1} \tag{13.73}$$

$\Delta\varphi_1$: Angle by which the perihelion advances after one revolution.

Here we used again that $\frac{3}{2} r_s A \ll 1$ is very small.

8 | It is convention to express $\Delta\varphi_1$ in terms of the parameters of the (Newtonian) elliptical orbit (13.51):

$$r = \frac{\ell}{1 + e\cos\varphi} \quad \text{with} \quad \ell := \frac{1}{A} \tag{13.74}$$

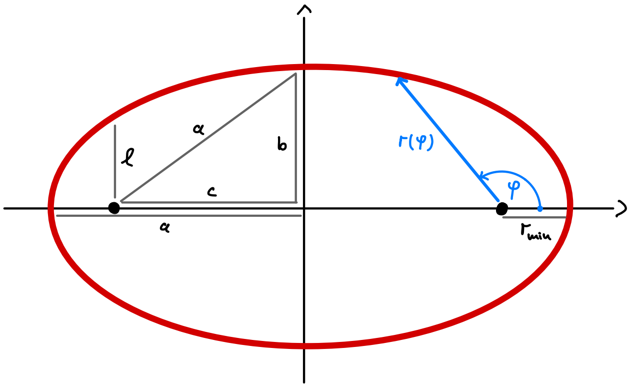
with …

- ↓ *eccentricity* e,
- ↑ *semi-latus rectum* ℓ,
- ↓ *perihelion distance* $r_{\mathrm{min}} := \frac{\ell}{1+e} = a(1-e)$, and
- ↓ *semi-major axis* a.

Eq. (13.73) →

$$\boxed{\Delta\varphi_1 = 3\pi\frac{r_s}{\ell} = \frac{3\pi}{1+e}\frac{r_s}{r_{\mathrm{min}}} = \frac{6\pi GM}{c^2 r_{\mathrm{min}}(1+e)} = \frac{3\pi r_s}{a(1-e^2)}} \tag{13.75}$$

- If a (b) is the semi-major (semi-minor) axis of an ellipse, and the focal points are at $|c| := \sqrt{a^2 - b^2}$ on the x-axis, the eccentricity is given by $e := \frac{c}{a} = \sqrt{1 - b^2/a^2}$. In our context, the perihelion distance is then $r_{\text{min}} := a - c = a(1-e)$. The parameter ℓ in the polar representation is given by $\ell := r_{\text{min}}(1+e) = a(1-e^2) = \frac{b^2}{a}$ and called ↑ *semi-latus rectum*:



- For Mercury we have $r_{\text{min}} \approx 4.6 \times 10^{10}$ m and $e \approx 0.206$, and the Schwarzschild radius of the Sun is $r_s \approx 2952$ m. Plugging in the numbers in Eq. (13.75) yields $\Delta\varphi_1 \approx 0.103''$, which is extremely small and not measurable. However, for each revolution around the Sun this shift accumulates, so that the effect amplifies over time. This is why the perihelion precession is typically measured in angular advance per 100 years. The orbital period of Mercury is $T \approx 0.241$ yr which leads to $N \approx 415$ revolutions per century. The prediction of GENERAL RELATIVITY for the perihelion advance of Mercury is then:

$$(\Delta\varphi)_{\text{Theory}} \approx 43.0'' \text{ per century} \tag{13.76}$$

Subtracting all known Newtonian effects (mostly due to planetary perturbations, this contribution is $\sim 532''$ per century, i.e., much larger than the relativistic effect) from the observed precession results in an unexplained difference of [215]

$$(\Delta\varphi)_{\text{Observation}} = (42.56 \pm 0.94)'' \text{ per century} \tag{13.77}$$

(a more recent analysis can be found in [216]). To comment this result in Einstein's words [13]:

> *Die Rechnung liefert für den Planeten Merkur ein Vorschreiten des Perihels um 43″ in hundert Jahren, während die Astronomen 45″ ± 5″ als unerklärten Rest zwischen Beobachtungen und Newtonscher Theorie angeben.*
>
> *Dies bedeutet volle Übereinstimmung.*

- Since the general relativistic perihelion precession scales with $\frac{r_s}{r_{\text{min}}}$ and accumulates with each revolution around the Sun, it is understandable that the effect was first observed for Mercury, the planet with the smallest perihelion distance (~ 0.31 AU) and the shortest orbital period (~ 0.24 yr). Nowadays one can measure the (considerably smaller) effect also for Venus and Earth.

- Compare this to your results of ➔ Problemset 1 where you studied relativistic, non-metric, linear theories of gravity:

  For the *scalar theory* you found

$$(\Delta\varphi_1)_{\text{Scalar}} = -\frac{2\pi GM}{c^2 r_{\text{min}}(1+e)}, \tag{13.78}$$

  which has both a wrong sign and a wrong prefactor compared to Eq. (13.75).

  For the *linear tensor theory* you found

$$(\Delta\varphi_1)_{\text{Tensor}} = \frac{8\pi GM}{c^2 r_{\text{min}}(1+e)}, \tag{13.79}$$

which has now the correct sign but still a wrong prefactor compared to Eq. (13.75).

These comparisons explain why we claimed that these theories make wrong predictions.

- Compare Eq. (13.75) with Einstein's result in Ref. [13] [Eq. (13) on p. 838]. Note that Einstein did not (and could not) know about the Schwarzschild solution at the time; he therefore employed approximate techniques to construct an appropriate metric. Since we also made approximations in the same order, the results coincide.

13.2.2. Deflection of light

We now study the second of Einstein's classical tests of GENERAL RELATIVITY: the deflection of light in the gravitational field of heavy bodies.

1 | Light rays follow *null* geodesics [← Eq. (11.5)]

We could now plug the Christoffel symbols of the Schwarzschild spacetime Eq. (13.10) [together with Eq. (13.20)] into the geodesic equation Eq. (10.131) and solve it for *light-like/null* trajectories (← Eq. (11.5)).

2 | There is a simpler method, though:

We can exploit our results for *massive* test particles in Section 13.2.1 to directly obtain the differential equation that describes null geodesics in the $\theta = \frac{\pi}{2}$ plane. The trick is that the geodesics of these particles must continuously morph into the null geodesics of light rays in the limit $m \to 0$ (for constant momentum).

$\sphericalangle$ Eqs. (13.46) and (13.57): $l = hm = mr^2\dot{\varphi} \to$ Orbital angular momentum

Light (photons) has momentum ($p = \hbar k$) but no mass ($m = 0$).

$$\to \lim_{m\to 0} l = \lim_{m\to 0} hm \overset{!}{>} 0 \to \lim_{m\to 0} h = \infty \to \lim_{m\to 0} A = \lim_{m\to 0} \frac{GM}{h^2} = 0$$

With this limit we find:

$$\text{Eq. (13.62)} \xrightarrow{m\to 0} u'' + u = \frac{3}{2} r_s u^2 \quad \text{with} \quad u = \frac{1}{r} \quad \text{and} \quad u'' = \frac{\mathrm{d}^2 u}{\mathrm{d}\varphi^2} \tag{13.80}$$

3 | Solution:

We solve Eq. (13.80) perturbatively along the same lines as in Section 13.2.1:

i | $\sphericalangle$ Homogeneous/linear part of Eq. (13.80):

$$u_0'' + u_0 = 0 \quad \Rightarrow \quad u_0 = \frac{1}{r} = \frac{1}{b}\sin(\varphi - \varphi_0) \tag{13.81}$$

Here, b and φ_0 parametrize the initial state.

We set *w.l.o.g.* $\varphi_0 = 0$ in the following. (Because of rotation symmetry.)

$\sphericalangle$ u_0 in Cartesian coordinates: (Note that $b = r\sin\varphi = \text{const}$ for the solution u_0.)

$$\vec{x}(\varphi) \equiv \begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix} \overset{13.81}{=} \begin{pmatrix} b\cot\varphi \\ b \end{pmatrix} \tag{13.82}$$

→ "Straight line" with ⁂ *impact parameter* b

The solution describes a horizontal line parallel to the x-axis that goes from $x = +\infty$ for $\varphi = 0$ to $x = -\infty$ for $\varphi = \pi$ and passes by the origin (where the Sun would be) at distance b.

¡! The trajectory u_0 does *not* solve the geodesic equation Eq. (13.80) of the Schwarzschild spacetime; it is therefore not a "straight line" (= autoparallel curve) in this metric. Only in the special case where $M = 0 \Rightarrow r_s = 0$ (i.e., when the Sun is gone) does u_0 describe the trajectory of light rays correctly. This is consistent as in this situation spacetime is flat and we would expect light to follow straight lines in Cartesian coordinates (which one can choose globally on a flat spacetime).

ii | Perturbative equation:

We plug in the unperturbed solution u_0 for the non-linear perturbation in Eq. (13.80):

$$u'' + u \approx 3\frac{r_s}{2b^2}\sin^2\varphi \tag{13.83}$$

$\overset{\circ}{\rightarrow}$ Particular solution:

$$u_1 = \frac{r_s}{2b^2}\left(1+\cos^2\varphi\right) \tag{13.84}$$

iii | → First-order solution:

$$u = \frac{1}{r} \approx u_0 + u_1 = \frac{1}{b}\sin\varphi + \frac{r_s}{2b^2}\left(1+\cos^2\varphi\right) \tag{13.85}$$

4 | We can again introduce spatial "Cartesian" coordinates: →

$$y = r\sin\varphi \overset{13.85}{=} b - \frac{r_s}{2b}r\left(2\cos^2\varphi + \sin^2\varphi\right) \overset{13.82}{=} b - \underbrace{\frac{r_s}{2b}\frac{2x^2+y^2}{\sqrt{x^2+y^2}}}_{\text{New! Cf. Eq. (13.82)}} \tag{13.86}$$

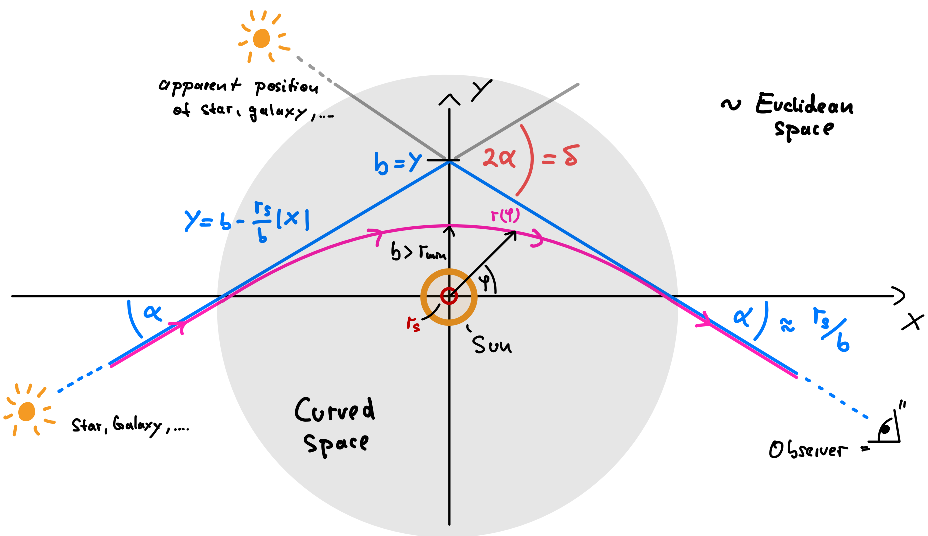
→ No longer "straight lines" in "Cartesian" coordinates!

5 | ∢ Asymptotic behavior for $x \to \pm\infty$:

$$y \approx b - \frac{r_s}{b}|x| \tag{13.87}$$

Do not forget the absolute value when evaluating $x^2/\sqrt{x^2}$!

This result an be illustrated as follows:



→ Deflection angle: (use the small angle approximation $\tan\alpha \approx \alpha$)

$$\delta = 2\alpha \overset{\alpha\ll 1}{\approx} 2\frac{r_s}{b} = \frac{4GM}{c^2 b} \overset{\text{Sun}}{\approx} \frac{1.75''}{b/R_\odot} \tag{13.88}$$

Here $R_\odot$ denotes the radius of the Sun.

- ¡! We established above that for $M = 0$ we can use Cartesian coordinates to describe the trajectory of the light ray on flat Minkowski space. Once we switch on gravity ($M \neq 0$), we can of course still use the coordinates (x, y), defined via Schwarzschild coordinates (r, φ) in the usual way. However, we cannot simply assume that they continue to have *metric meaning* (i.e., are *Cartesian*)! (Recall that r has no direct metric meaning in the Schwarzschild geometry either [← Eq. (13.33)].) Thus the fact that Eq. (13.86) no longer describes a "straight line" in (x, y) *coordinate space* has no physical interpretation a priori.

  Luckily, we usually observe celestial bodies from *far away* (and the light deflected by them reaches them from far away). Since we know that the Schwarzschild metric induced by these objects is asymptotically flat, we can study the light long before and after it entered the gravitational field of these objects. In these regions, spacetime is approximately Minkowskian, and the coordinates (x, y) are approximately Cartesian (they are spatial components of an inertial coordinate system which, as discussed in Section 1.1, carries metric information). Since the angle Eq. (13.88) is defined between two straight lines in this region of space, it has physical meaning and observable effects (→ *below* and Section 13.2.3).

- The predicted deflection of light that passes close by the Sun ($b \sim R_\odot \Rightarrow \delta \approx 1.75''$) was first measured by ARTHUR EDDINGTON and collaborators during their famous expedition to West Africa (Príncipe) and Brazil (Sobral) [217], where they exploited the solar eclipse on 29. May 1919 to observe stars that are visible close to the solar disk only when it is covered by the moon (↑ *Eddington experiment*). In their paper, they distinguish three possible outcomes: (1) light is not deflected by gravity, (2) light is deflected by the Newtonian angle $\delta/2 \approx 0.87''$ (→ *below*), or (3) light is deflected by the angle $\delta \approx 1.75''$ predicted by GENERAL RELATIVITY. They summarize their meticulous analysis as follows (p. 332):

  > *Thus the results of the expeditions to Sobral and Principe can leave little doubt that a deflection of light takes place in the neighbourhood of the Sun and that it is of the amount demanded by* EINSTEIN'S *generalised theory of relativity, as attributable to the Sun's gravitational field.*

  This result mad headlines all over the world, contributed to the wide acceptance of GENERAL RELATIVITY, and catapulted Einstein to fame.

  For a review of various experimental results (up to 1960) regarding the deflection of light see Ref. [218]. The precision of the Eddington experiment was rather low (and its significance later debated, see Ref. [219] for a review). However, later variations of the experiment that used radio waves instead of light verified the predictions of GENERAL RELATIVITY to very high precision. Ref. [220], for example, reports only a deviation of $\delta_{\text{measured}}/\delta_{\text{predicted}} = 1.007 \pm 0.009$ from the predictions of GENERAL RELATIVITY.

- In the aftermath of establishing SPECIAL RELATIVITY, Einstein studied uniformly accelerated frames of reference and already proposed the equivalence principle, equating uniform acceleration with uniform gravitational fields. This led him 1907 to the prediction that light must be deflected by gravity. He states in Ref. [95] (p. 212):

  > *Es folgt hieraus, daß die Lichtstrahlen, […], durch das Gravitationsfeld gekrümmt werden; […]*

Later, in 1911, Einstein elaborated on this idea in his paper *"Über den Einfluß der Schwerkraft auf die Ausbreitung des Lichtes"* [103] and predicted the deflection angle

$$\delta_{\text{Newton}} = \frac{2GM}{c^2 b} = \frac{\delta}{2}\,, \tag{13.89}$$

which is exactly *half* the prediction (13.88) of GENERAL RELATIVITY. He evaluates it for a light ray that skims the Sun and concludes (p. 908):

> *Ein an der Sonne vorbeigehender Lichtstrahl erlitte demnach eine Ablenkung vom Betrage* $4\cdot 10^{-6} = 0.83$ *Bogensekunden.*

The result (13.89) can be obtained by postulating that Newtonian gravity also affects light rays because, according to SPECIAL RELATIVITY, photons have a "dynamical mass" $m = E/c^2 = h\nu/c^2$. Due of the universality of free fall, the trajectory of a particle that shoots by the Sun (and is on an unbound trajectory) only depends on its initial velocity and position (and not its mass). It is then reasonable to postulate that the same trajectories are followed by photons with c as initial velocity. This purely Newtonian calculation (with appropriate approximations) yields the deflection angle Eq. (13.89).

In the course of completing GENERAL RELATIVITY in 1915, Einstein realized that the actual deflection angle predicted by GENERAL RELATIVITY is *twice* his original prediction of 1911. He presented his results in a meeting of the Prussian Academy of Science on 25. March 1915 [221] and published the calculation of the correct deflection angle (13.88) in his famous 1916 paper [20] (which sums up the results accumulated during 1915).

Fun fact: While the deflection angle $1.7''$ is correctly stated in Ref. [20], the corresponding equation (74) on page 822 is actually off by the important factor of 2 due to a printing error; it *should* read $B = \frac{2\alpha}{\Delta} = \frac{\kappa M}{2\pi\Delta}$ with $\kappa = \frac{8\pi K}{c^2}$, where K denotes Newton's gravitational constant, B is the deflection angle, and Δ the impact parameter [222].

The difference between Newtonian and generally relativistic predictions of the deflection angle can be traced back to the curvature of *space* that is missing in the former (Newtonian space is Euclidean) and included in the latter [due to the factor $(1 - r_s/r)^{-1}$ for $\mathrm{d}r$ in the Schwarzschild metric (13.25), recall Eq. (13.33) ff]. Note that because of Eqs. (11.64) and (11.65), the prefactor $(1 - r_s/r)$ of $\mathrm{d}t$ in (13.25) is responsible for reproducing Newtonian physics; it is then the additional prefactor $(1 - r_s/r)^{-1}$ of $\mathrm{d}r$ that is responsible for doubling the deflection angle in the Schwarzschild metric.

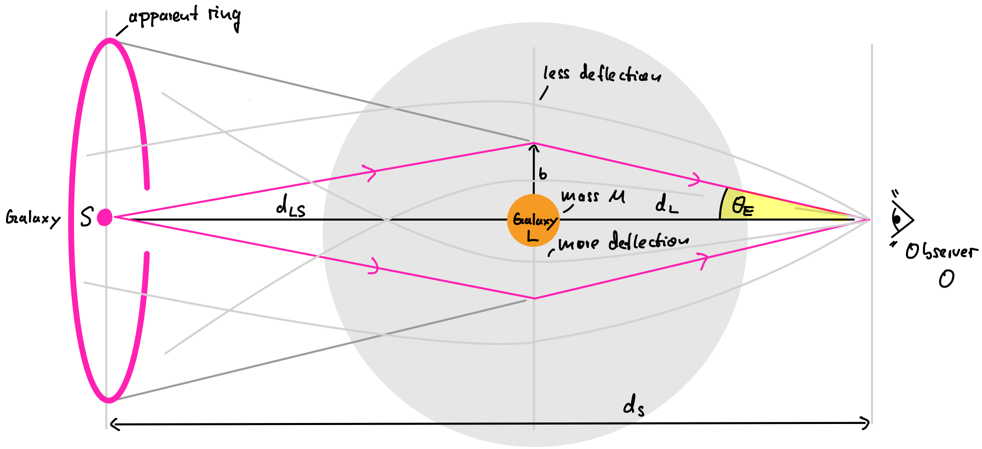
13.2.3. Gravitational lensing

A direct consequence of the deflection of light is that large masses can act as "lenses" for distant observers:

6 | Example: Here we consider the most symmetric (and rarest) scenario:

$\triangleleft$ *Collinear* constellation with ...

- light source S (e.g., a galaxy),
- heavy mass/lens L (e.g., another galaxy),
- observer O (a telescope on Earth).



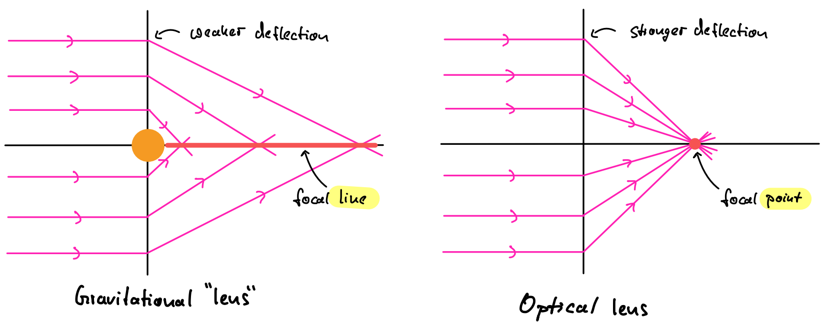
Axial symmetry → Point source appears *ring-shaped* → ⁂ *Einstein ring*

→ Straightforward linearized trigonometry leads to the angular size of the Einstein ring:

$$\theta_E \stackrel{\circ}{=} \sqrt{\frac{4GM}{c^2} \cdot \frac{d_{LS}}{d_L d_S}} \qquad \text{⁂ } \textit{Einstein angle} \tag{13.90}$$

For details: ↑ Carroll [102] (§8.6, p. 349 ff.).

- ¡! Because of the $1/b$ dependence of the deflection angle (13.88), gravitational lenses do not have a focal point but a focal *line* (along the optical axis, behind the lens). Thus, strictly speaking, gravitational lenses are no lenses:



- If source and/or observer are slightly off-axis, the Einstein ring typically breaks into two copies of the imaged object. If the alignment is almost perfect, the ring can morph into a "horseshoe Einstein ring", as shown in Fig. 13.1 (a). When the lens breaks the rotational symmetry (think of an elongated galaxy), the image can consist of four copies of the same object, called an ↑ *Einstein cross* [Fig. 13.1 (b)]. Typical constellations are even less symmetric and produce a warped mess, as shown in Fig. 13.1 (c).
- That massive bodies can act as "gravitational lenses" was discussed by Einstein in 1936 [223]. Since Einstein considered *stars* as lenses, he came to the conclusion that the effect was way too small to be observable:

  > *Therefore, there is no great chance of observing this phenomenon, even if dazzling by the light of the much nearer star B is disregarded.*

  However, one year later, Fritz Zwicky suggested that *galaxies* might be massive enough to cause observable lensing effects [224]. The first gravitational lens (indeed caused by a galaxy) was then observed in 1979 [225] (the ↑ *Twin Quasar*, a single quasar that appears twice due to a gravitational lens). The first complete Einstein ring was observed later (in 1997) by the Hubble telescope in the infrared [226].

Nowadays, a plethora of gravitational lenses have been identified (→ Fig. 13.1).

7 | Observations:

Here a few examples of observed gravitational lenses:

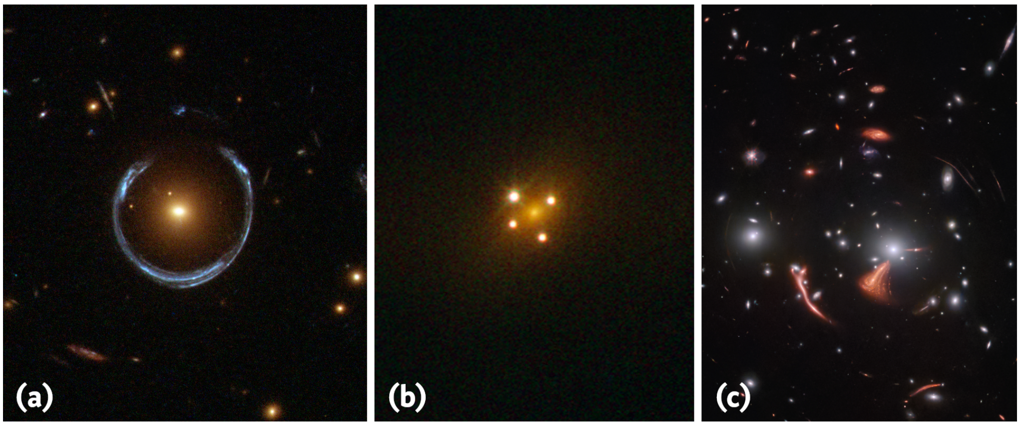


FIGURE 13.1. • **Gravitational lenses: (a)** A horseshoe Einstein ring photographed by Hubble in 2011: *"The gravity of a luminous red galaxy in the foreground has gravitationally distorted the light from a much more distant blue galaxy."* [227] **(b)** An ↑ *Einstein cross* photographed by Hubble in 2012: *"The foreground galaxy's gravity acts as a lens that bends and amplifies the light from a quasar behind it, producing four images of the distant object."* [228] **(c)** A large gravitational lens photographed by the James Webb Space Telescope in 2023: *"A galaxy cluster in the foreground has magnified distant galaxies, warping their shapes and creating the bright smears of light spread throughout this image."* [229]

8 | Applications:

Nowadays, gravitational lensing is used as a *tool* in astronomy:

For more details: ↑ Ref. [230].

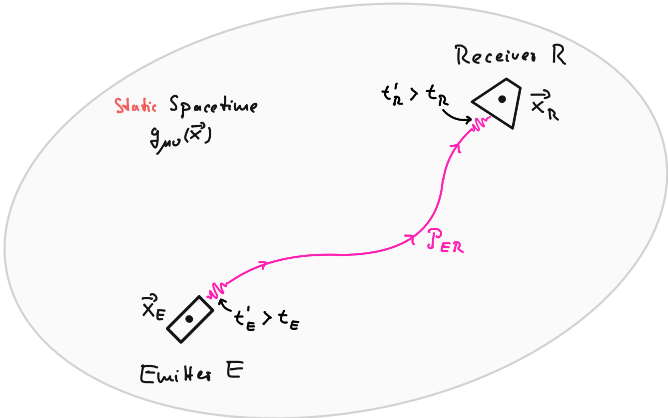
- ↑ *Microlensing:* Gravitational lensing of background light sources by small, mostly invisible objects (like exoplanets, neutron stars, black holes ...) can be used to detect and study them. These objects are too light to cause observable distortions of the image; however, lensing also changes the apparent *brightness* of the background object – and changes in brightness over time can be detected even if the lensing itself cannot be resolved.
- ↑ *Weak lensing:* The lensing that produces Einstein rings and multiple images of the same object is called ↑ *strong lensing* (and is quite rare). In most directions of space, there are no observable strong lensing phenomena. By contrast, *weak lensing* describes the slight and ubiquitous "warping" of background sources by the foreground mass distribution. This warping can be used statistically to gain information about the (often invisible) mass distribution in the foreground [recall the blue ↑ *lensing map* used to study the ← *Bullet cluster* in Fig. 12.1 (c)].
- Since strong gravitational lenses can magnify extremely distant objects, it has been hypothesized to use the Sun as a "telescope" [231]. The problem with this proposal is that the nearest point on the half-infinite focal line of the Sun is about 550 AU (astronomical units = Sun-Earth distances) away – and this is where a space-borne observatory would have to be in order to use the Sun as a lens. For comparison, Voyager 1 is with only $\sim$ 164 AU the most distant spacecraft we managed to deploy.

13.2.4. Gravitational redshift

Based on the conservation of energy, and the possibility to create and annihilate particles from and into photons (reflecting the equivalence of mass an energy), we already concluded in Section 8.3 that the wavelength of light that escapes a gravitational potential must increase, i.e., the light must be *redshifted*. Here we finally confirm this prediction within the full framework of GENERAL RELATIVITY:

1 | ⊲ *Stationary* emitter E at $\vec{x}_E$ and receiver R at $\vec{x}_R$ in a *static* metric:

Definition of a static metric: ← Eq. (11.136) in Section 11.5.



The following derivation does not rely on the Schwarzschild metric; we will specialize to this particular metric later.

2 | ⊲ Light signal emitted by E at t_E and received by R at t_R:

The light follows a *light-like* trajectory. $\rightarrow \mathrm{d}s^2 = 0 \xrightarrow{\text{Eq. (11.136)}}$

$$ct_R - ct_E = c\int_{t_E}^{t_R} \mathrm{d}t = \int_{\mathcal{P}_{ER}} \underbrace{\sqrt{\frac{-g_{ij}}{g_{00}}\mathrm{d}x^i\,\mathrm{d}x^j}}_{\text{Independent of } t} \tag{13.91}$$

$\mathcal{P}_{ER}$ is the spatial path followed by the light signal from $\vec{x}_E$ to $\vec{x}_R$.

3 | ⊲ *Second* light signal from E to R:

Metric *static* → Signal follows the *same* path $\mathcal{P}_{ER} \xrightarrow{\text{Eq. (13.91)}}$

$$\underbrace{ct_R - ct_E}_{\text{first signal}} = \underbrace{ct'_R - ct'_E}_{\text{second signal}} \quad\Leftrightarrow\quad \Delta t_R := t'_R - t_R = t'_E - t_E =: \Delta t_E \tag{13.92}$$

This means that the *coordinate* time differences between the first and the second signal are the same for both emitter and receiver!

Assume that at the first signal a laser is switched *on*, and on the second signal it is switched *off*.

Let there be n oscillations of the electromagnetic field emitted by E and received by R →

$$\frac{n}{\Delta t_R} \overset{13.92}{=} \frac{n}{\Delta t_E} \tag{13.93}$$

¡! This is the *coordinate* frequency of the light at R and E, not the measured frequency!

4 | Proper time $\mathrm{d}\tau = c^{-1}\mathrm{d}s$ measured at the position of E and R:

$$\Delta\tau_{E/R} \overset{11.136}{=} \sqrt{g_{00}(\vec{x}_{E/R})}\,\Delta t_{E/R} \tag{13.94}$$

Recall that we assume E and R to be stationary in the chosen coordinates $x^\mu = (ct, \vec{x})$.

→ With this we find for the *measured* frequencies of the emitted and received light:

$$\frac{\nu_R}{\nu_E} = \frac{n/\Delta\tau_R}{n/\Delta\tau_E} \overset{13.94}{=} \sqrt{\frac{g_{00}(\vec{x}_E)}{g_{00}(\vec{x}_R)}} \cdot \frac{n/\Delta t_R}{n/\Delta t_E} \overset{13.93}{=} \sqrt{\frac{g_{00}(\vec{x}_E)}{g_{00}(\vec{x}_R)}} \tag{13.95}$$

5 | So far we only used that the metric is static; now we specialize to the metric of a spherical mass:

⊲ Schwarzschild metric Eq. (13.25) →

$$\frac{\nu_R}{\nu_E} = \sqrt{\frac{1 - r_s/r_E}{1 - r_s/r_R}} \quad \text{or} \quad 1 + z := \frac{\lambda_R}{\lambda_E} = \sqrt{\frac{1 - r_s/r_R}{1 - r_s/r_E}} \tag{13.96}$$

with ⁂ *Redshift parameter* z

For $r_R > r_E$ it follows $\lambda_R > \lambda_E \;\Leftrightarrow\; z > 0 \;\rightarrow$ ⁂ *Gravitational redshift*

- The gravitational redshift was first experimentally probed and verified by Robert Pound and Glen Rebka in 1960 with their famous ↑ *Pound-Rebka experiment* [104, 105]. The experiment was conducted in a laboratory on Earth and exploited the extremely high spectral resolution provided by the ↑ *Mößbauer effect*.
- The gravitational redshift also affects photons emitted by the Sun and received on Earth. This particular probe of the redshift has been successful as well [232], but is complicated by the motion of the emitting atoms on the Sun (which causes random Doppler shifts).

6 | Approximations:

- In many situations it is $\frac{r_s}{r} \ll 1$ → Newtonian approximation:

$$1 + z \overset{\circ}{\approx} 1 + \frac{r_s}{2}\left(\frac{1}{r_E} - \frac{1}{r_R}\right) \tag{13.97}$$

To show this, expand Eq. (13.96) in first order of r_s/r_E and r_s/r_R.

Let $r_R = r_E + \Delta h$ with height difference $\Delta h \ll r_E$ →

$$1 + z \overset{\text{Taylor}}{\approx} 1 + \frac{r_s}{2} \cdot \frac{\Delta h}{r_E^2} = 1 + \underbrace{\frac{GM}{r_E^2}}_{=g} \cdot \frac{\Delta h}{c^2} = 1 + \frac{g\Delta h}{c^2} \tag{13.98}$$

Here g is the gravitational acceleration at the emitter (and receiver, since $\Delta h \ll r_E$).

→ Same result as Eq. (8.14) in Section 8.3 ☺

We therefore confirmed our previous derivation in the Newtonian limit. The fact that light is affected by gravity (and redshifted if it leaves a gravitational potential) is therefore not a consequence of the particular structure of the Einstein field equations (we didn't now about them in Section 8.3), but follows from the principles of SPECIAL RELATIVITY, together with the EEP of GENERAL RELATIVITY.

This explains how Einstein could predict the gravitational redshift in 1907 (p. 209 of Ref. [95]) without knowing about curved spacetime. However, this approach is only applicable to *homogeneous* gravitational fields (which is often justified on Earth). The exact value of the redshift for light that traverses large distances (and thereby probes the non-homogeneity of gravitational fields), and/or comes close to the Schwarzschild radius, can only be computed with the machinery of GENERAL RELATIVITY as applied above (including the EFEs).

- In astronomical scenarios, the emitters are often excited atoms close to the surface of a star, so that $r_E = R_*$ with R_* the radius of the star. Telescopes on Earth are the receivers, so that usually $r_R \to \infty$ is a good approximation. One then finds for the ← *redshift parameter* z:

$$1 + z \overset{13.96}{\approx} \left(1 - \frac{r_s}{R_*}\right)^{-\frac{1}{2}} \tag{13.99}$$

Such redshifts can be measured by spectroscopy since we know the optical transitions of the elements that serve as emitters (e.g., hydrogen). Spectroscopic analysis of the light emitted by a star then reveals the redshift by comparison with the wavelength one would measure for the same elements in a laboratory on Earth.

Beware: The situation is significantly complicated by various other phenomena that can change the wavelength of light. For example, relative motion leads to the ↓ *Doppler effect*. Furthermore, the metric of our universe is *not* a static Schwarzschild metric but describes an expanding spacetime. This leads to an additional ↑ *cosmological redshift* that depends on the time the light requires to reach us.

13.2.5. Gravitational time dilation

The gravitational time dilation Eq. (13.94) causes the gravitational redshift discussed in Section 13.2.4. We also covered it in our discussion of the role played by the Schwarzschild time coordinate [← Eq. (13.27)]. So all important mathematical results have already been stated before:

7 | ⊲ Two stationary clocks A/B located at $\vec{x}_i = (r_i, \theta_i, \varphi_i)$ $(i = A, B)$:

The clocks are stationary *in Schwarzschild coordinates*.

Measure proper time between the same *coordinate* time slices t and $t + \Delta t$ $\xrightarrow{\text{Eq. (13.94)}}$

$$\boxed{\frac{\Delta\tau_A}{\Delta\tau_B} \overset{13.94}{=} \sqrt{\frac{g_{00}(\vec{x}_A)}{g_{00}(\vec{x}_B)}} \overset{13.25}{=} \sqrt{\frac{1 - r_s/r_A}{1 - r_s/r_B}}} \tag{13.100}$$

$r_B > r_A \Rightarrow \Delta\tau_B > \Delta\tau_A \rightarrow$ ⁂ *Gravitational time dilation*

To understand why and how B "sees" A tick slower: → *below*.

- Like the gravitational redshift, gravitational time dilation is not a probe for the validity of the Einstein field equations (at least if applied in the weak field limit) but of the EEP. That is, the effect can be derived from assuming the validity of SPECIAL RELATIVITY together with the equivalence principle.

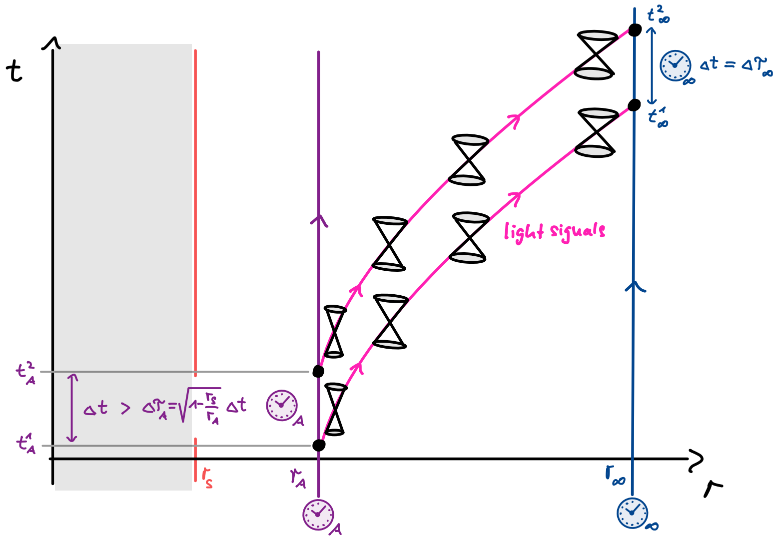
This explains how Einstein could predict the gravitational time dilation already in 1907 (p. 208-209 of Ref. [95]).

- Recall that special relativistic time dilation (← Section 2.2) is *symmetric* in that two (inertial) observers both measure the clock of the other tick slower. This symmetric effect is somewhat artificial because it relies on the comparison of *different clocks* – which makes apparent the relativity of simultaneity. To compare two clocks that travel different paths *twice*, at least one had to be accelerated (in Minkowski space!), recall our discussion of the twin "paradox" in Section 2.4. In this scenario, the effect was no longer symmetric and both observers agreed on their relative time delay. Gravitational time dilation is a generalization of this phenomenon to curved spacetime. It is also asymmetric in that an Earth-bound observer sees the clock of an asymptotically distant observer run *faster*, whereas this observer sees the Earth-bound clock run *slower*.
- The slowdown of time becomes extreme if one approaches the Schwarzschild radius. As already discussed, this is impossible for "normal" objects like planets and stars, which is why the scenario is irrelevant for physics in the solar system. However, if we *could* approach the event horizon of a black hole (we don't have to reach it, being nearby $r \gtrsim r_s$ is enough), the effect of gravitational time dilation can become arbitrarily large.

  Fun fact: This effect is one of the main plot points of the 2014 movie INTERSTELLAR. In the movie, the protagonists land on "Miller's planet" – a planet that orbits a supermassive black hole – where *one hour* proper time corresponds to *seven years* proper time at $r \to \infty$ (e.g., on Earth). If you stay too long (say one day) and fly back to Earth, everyone you knew will be long dead ☹.

8 | To understand how stationary observers at different locations (in Schwarzschild coordinates) "see" clocks tick, consider the following setup:

(This is the sketch from Section 13.1.3, reprinted for your convenience.)



Let A be a stationary clock in the gravitational field at r_A and B a clock at spatial infinity ("far away"): $r_B \to r_\infty = \infty$. Assume A sends the reading of its clock at coordinate times t_A^1 and t_A^2 with radio signals to B. Because the Schwarzschild metric is static, the spacetime trajectories of the two signals are congruent, so that the coordinate time differences Δt between the two signals are the same for A and B [see sketch, mathematically this follows from Eq. (13.92)].

But the two clock readings sent by A differ by

$$\Delta\tau_A = \sqrt{1 - \frac{r_s}{r_A}}\,\Delta t = \sqrt{1 - \frac{r_s}{r_A}}\,\Delta\tau_\infty < \Delta\tau_\infty\,, \tag{13.101}$$

which is less than the time $\Delta\tau_\infty$ elapsed for B between the two messages.

→ B concludes that the clock A runs slower!

9 | Weak-field approximation:

⊲ Weak-field limit $\frac{r_s}{r} \ll 1$: Eq. (13.100) $\xrightarrow{\text{Eq. (13.97)}}$

$$\frac{\Delta\tau_A}{\Delta\tau_B} - 1 \approx \frac{r_s}{2}\left(\frac{1}{r_B} - \frac{1}{r_A}\right) \tag{13.102}$$

→ Relative tick rate:

$$\frac{\Delta T}{T} \equiv \frac{\Delta\tau_A - \Delta\tau_B}{\Delta\tau_B} \overset{13.102}{\approx} \frac{r_s}{2}\left(\frac{1}{r_B} - \frac{1}{r_A}\right) \overset{13.98}{\approx} \frac{g\Delta h}{c^2} \tag{13.103}$$

with gravitational acceleration $g = \frac{GM}{r_B^2}$.

Here we used $r_A = r_B + \Delta h$ with height difference $\Delta h \ll r_B$ as in Eq. (13.98).

If we use $g = 9.81\,\mathrm{m}^2/\mathrm{s}$ and the height of Mount Everest $\Delta h = 8848\,\mathrm{m}$, we find $\Delta T/T \sim 10^{-12}$ for the relative frequency difference between a clock on sea level and one on the summit. This variation is small but well within the precision of modern atomic clocks (→ *below*).

10 | Experiments:

- The space-born ↑ *Gravity Probe A* experiment (1976) was one of the first to directly measure the effect of gravitational time dilation [233]. It confirmed the prediction of GENERAL RELATIVITY to high precision.
- For the ← *Hafele-Keating experiment* (1971), the gravitational time dilation due to the difference in height between the airplanes and the ground-based reference clock had to be taken into account to match observation and theory [46, 47]; recall ➲ Problemset 5.
- Modern ↑ *optical atomic clocks* are precise enough to directly measure the gravitational time dilation simply by lifting them a few centimeters [234]:

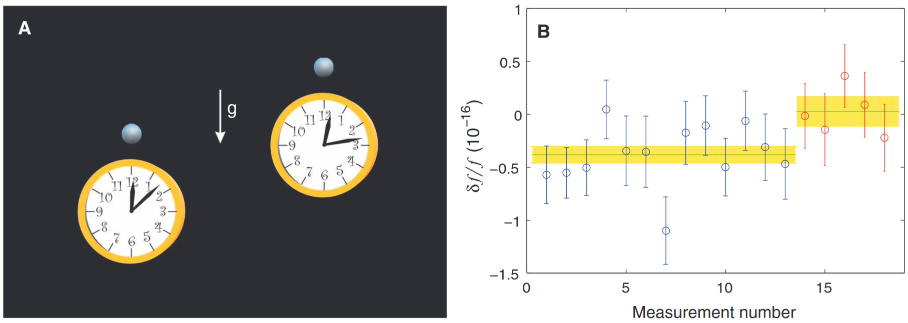


FIGURE 13.2. • Gravitational time dilation measured by optical clocks: In 2010, the precision of optical clocks (a modern variety of atomic clocks) reached levels that allowed for the direct verification of the gravitational time dilation by elevating one clock by 33 cm (between measurement numbers 13 and 14 in panel B) [234]. The frequency (tick rate) of the clock clearly increases, as predicted by GENERAL RELATIVITY.

- The recent development of smaller and more robust optical clocks gave birth to the new field of ↑ *relativistic geodesy*, i.e., the mapping of Earth by using optical clocks to measure *heights* by proxy of gravitational time dilation (see Ref. [235] and references therein). Note that gravitational time dilation does not actually measure heights (with respect to whatever) but the *gravitational potential*. This means that the tick rate of your clock also changes when you are above a geological anomaly with higher/lower average mass density (which is also valuable information):

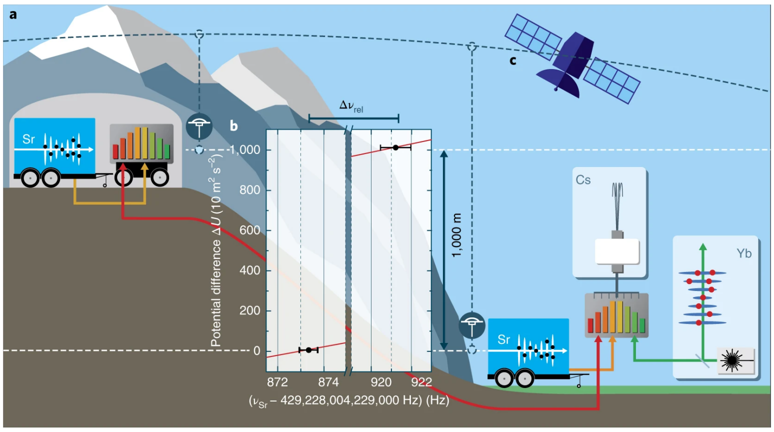


FIGURE 13.3. • Relativistic Geodesy: By now, optical clocks have become small enough so that one can use them *to measure hights* [235]; this establishes the field of ↑ *relativistic geodesy*: the measurement of differences in the gravitational potential by exploiting generally relativistic effects and our technological ability to measure times extremely precisely.

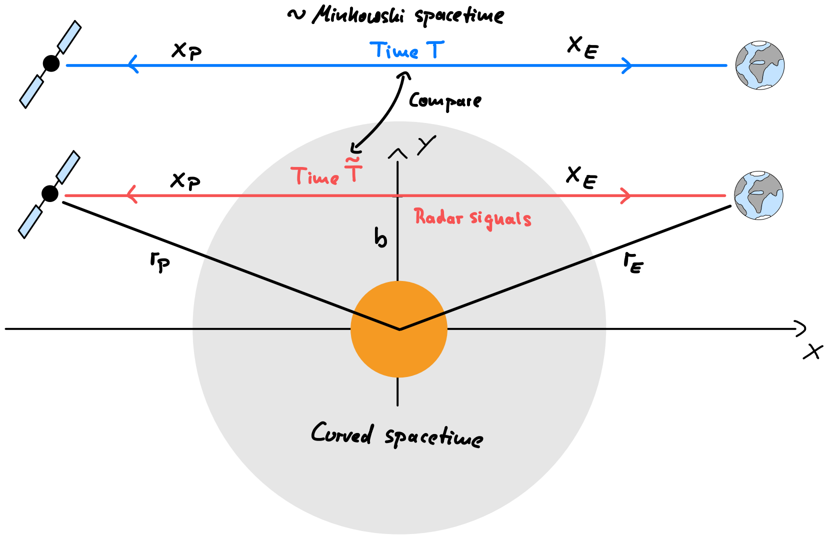
- Famously, both special relativistic (← Section 2.2) and gravitational time dilation are relevant effects that must be taken into account for the ↑ *Global Positioning System (GPS)* to work. The system is based on a fleet of satellites equipped with atomic clocks that broadcast their time (plus additional data) to Earth; these timestamps can be used by Earth-bound receivers to calculate their position relative to (at least) three satellites. Special relativistic time dilation makes the clocks of the satellites run *slower* with respect to stationary clocks on Earth, whereas gravitational time dilation makes them run *faster*. For the orbit of GPS satellites, the gravitational time dilation dominates, so that their clocks run *faster* than Earth-bound clocks. This effect (among others) must be taken into account for the system to function; see Ref. [236] for details.

13.2.6. Shapiro time delay

Besides the three classical tests of GENERAL RELATIVITY (perihelion precession, deflection of light / lensing, gravitational redshift / time dilation), there is is a *fourth* test proposed 1964 by IRWIN SHAPIRO [237]: Light that travels through the gravitational field of a heavy mass takes a bit longer than it would without the mass:

1 | ∢ Radar signal bounced between Earth and satellite (or another planet):

For a strong effect, the satellite must be in (approximate) ↑ *superior conjunction* with Sun, such that the signal passes close to the Sun and experiences a strong gravitational field.



Question: How much time elapses on Earth during a round trip of the signal?

To simplify calculations, we make the following (justified) approximations:

- The deflection of the signal is small and can be neglected.
- The radar signal is fast so that we can consider all bodies as stationary.
- The time elapsed on Earth is assumed to be approximately Schwarzschild coordinate time.

2 | ∢ $\theta = \frac{\pi}{2}$ plane $\xrightarrow{\text{Eq. (13.25)}}$

$$ds^2 = \left(1 - \frac{r_s}{r}\right) d(ct)^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2 d\varphi^2 \overset{\text{Light}}{=} 0 \tag{13.104}$$

Light/radar ray follows (approximately) straight line $y = r \sin\varphi = b = \text{const} \xrightarrow{\circ}$

$$d(ct)^2 = \left[\left(1 - \frac{r_s}{r}\right)^{-2} + \left(1 - \frac{r_s}{r}\right)^{-1} \frac{b^2}{(r^2 - b^2)}\right] dr^2 \tag{13.105}$$

To show this use $0 = db = \sin\varphi\, dr + r\cos\varphi\, d\varphi$ and $\tan^2\varphi = b^2/(r^2 - b^2)$.

3 | Take root & expand in linear order of $\frac{r_s}{r} \xrightarrow{\circ}$

$$d(ct) \approx \frac{dr}{\sqrt{1 - \frac{b^2}{r^2}}} \left(1 + \frac{r_s}{r} - \frac{1}{2}\frac{r_s b^2}{r^3}\right) \tag{13.106}$$

4 | Integration along path $(x, y = b)$ for $x \in [-x_P, x_E] \overset{\circ}{\rightarrow}$

$$c\tilde{T} = \overbrace{\underbrace{x_P + x_E}_{\text{Euclidean distance}}}^{\equiv cT} + \underbrace{r_s \ln \frac{(r_P + x_P)(r_E + x_E)}{b^2} - \frac{r_s}{2}\left(\frac{x_P}{r_P} + \frac{x_E}{r_E}\right)}_{\text{Additional time delay due to gravity} \to \text{Shapiro delay}} \tag{13.107}$$

Since r is not single-valued on the trajectory from Earth to the satellite, one has to add up the integrals from r_E to $r = b$ (segment x_E) and from $r = b$ to r_P (segment x_P). To derive the result, use $x_i = \sqrt{r_i^2 - b^2}$ for $i = E, P$.

$2\tilde{T}$: (Coordinate) time for round trip of signal *with Sun* ($r_s > 0$)
$2T$: (Coordinate) time for round trip of signal *without Sun* ($r_s = 0$)

Ignoring the gravitational time dilation on Earth (due to the gravitational field of the Sun), $2\tilde{T}$ is approximately the time measured by a clock on Earth for a round trip of the signal.

5 | Let $\Delta T := \tilde{T} - T$ and assume $r_E, r_P \gg b$ so that $x_E \approx r_E$ and $x_P \approx r_P$ →

$$\Delta T \approx \frac{r_s}{c}\left(\ln \frac{4x_P x_E}{b^2} - 1\right) > 0 \quad \text{⁂ } \textit{Shapiro time delay} \tag{13.108}$$

→ Light travels *slower* in a gravitational field than in flat Minkowski space!

¡! This slowdown is "seen" by an observer at infinity (it is a slowdown in the *Schwarzschild coordinate velocity*); in every *local inertial frame* the speed of light remains constant (namely c, ← Section 11.1).

- The effect is more pronounced for smaller impact parameters b; this explains why approximate superior conjunction is needed to measure the effect.
- To be fully correct, one has to translate the *coordinate* time delay Eq. (13.108) via Eq. (13.94) into the *proper* time delay measured by clocks on Earth (using the Schwarzschild metric of the Sun). We omit this correction here because it is irrelevant for understanding the Shapiro effect qualitatively.
- To get a feeling for the magnitude of the effect, let us assume we bounce radar signals off Mercury ($x_P \approx 5.8 \times 10^{10}$ m) and receive them on Earth ($x_E \approx 14.9 \times 10^{10}$ m). The smallest impact parameter possible is the radius of the Sun: $b \approx 6.96 \times 10^8$ m. With the Schwarzschild radius $r_s \approx 3 \times 10^3$ m (of the Sun) one finds a one-way delay of $\Delta T \approx 102\,\mu\text{s}$, which is of course well within the capabilities of modern clocks.
- The relativistic time delay was proposed by Irwin Shapiro as a fourth test of general relativity in Ref. [237]. First results were obtained by reflecting radar signals off Venus and Mercury [238, 239] and confirmed the predictions of general relativity. The most precise measurement thus far was obtained by monitoring the radio link of the Cassini spacecraft; these measurements confirmed the predictions of general relativity with a relative deviation of only $\sim 2 \times 10^{-5}$ [240].