↓Lecture 27 [02.07.24]

13. Applications & Predictions

Now that the framework of GENERAL RELATIVITY is fully developed, we can start using it. As already mentioned, solving the Einstein field equations is hard, and in most realistic scenarios impossible. This is why we focus on the simplest and most symmetric settings – which still does not save us from mathematical complexity. Thus, instead of struggling with *conceptual* subtleties, we will mostly fight *technical* (mathematical) issues in this chapter.

¡! Studying applications and predictions of GENERAL RELATIVITY is a vast topic, deserving its own course. This chapter only scratches the surface of this multifaceted (and active) field of research.

A comprehensive review of experimental tests of GENERAL RELATIVITY can be found in Ref. [200].

13.1. The gravitational field of a spherical mass

The first (and most important) exact solution of the Einstein field equations was obtained by German physicist KARL SCHWARZSCHILD at the end of 1915, only a few weeks after Einstein published his field equations; the so called → *Schwarzschild metric* was published in January 1916 [201]. Schwarzschild found his solution while serving in the German army (during World War I); he died only a few months later in May 1916 (due to a disease he developed at the Russian front).

Here is how Schwarzschild sells his solution in Ref. [201] (§2):

> *Hr. EINSTEIN hat gezeigt, daß dies Problem [der sphärisch symmetrischen Massenverteilung] in erster Näherung auf das Newtonsche Gesetz führt [← Section 12.1.1] und daß die zweite Näherung die bekannte Anomalie in der Bewegung des Merkurperihels richtig wiedergibt [→ Section 13.2.1]. Die folgende Rechnung liefert die strenge Lösung des Problems. Es ist immer angenehm, über strenge Lösungen einfacher Form zu verfügen [recht hat er ☺]. Wichtiger ist, daß die Rechnung zugleich die eindeutige Bestimmtheit der Lösung ergibt, über die Hrn. EINSTEINS Behandlung noch Zweifel ließ, und die nach der Art, wie sie sich unten einstellt, wohl auch nur schwer durch ein solches Annäherungsverfahren erwiesen werden könnte. Die folgenden Zeilen führen also dazu, Hrn. EINSTEINS Resultat in vermehrter Reinheit erstrahlen zu lassen.*

Einstein was surprised that Schwarzschild succeeded so quickly in deriving an exact solution for his field equations. He writes on 29. December 1915 [202] in a letter to Schwarzschild:

> *Ihre Rechnung, die den Eindeutigkeitsbeweis für das Problem liefert, ist höchst interessant. Hoffentlich veröffentlichen Sie dieselbe bald! Ich hätte nicht gedacht, dass die strenge Behandlung des Punktproblems so einfach wäre.*

Approximately spherically symmetric masses are ubiquitous in our universe: think of planets, stars, and black holes. It is thus a reasonable first step, after setting up the Einstein field equations, to ask what the metric induced by spherically symmetric bodies looks like (in vacuum, outside of the mass itself), and which modifications of the dynamics of test particles moving in such gravitational fields GENERAL RELATIVITY predicts. In this section, we discuss this scenario in detail.

Here we consider only spherically symmetric mass distributions that are *non-rotating* and *uncharged*. In particular the first assumption is often not satisfied by objects in space (celestial bodies typically rotate); taking into account the *angular momentum* of masses leads to a (more complicated) cousin of the Schwarzschild metric, the so called ↑ *Kerr metric* (which we will not discuss here due to its complexity).

13.1.1. Spherically symmetric spacetimes

1 | Recall: Minkowski metric in spherical coordinates:

$$ds^2 = c^2 dt^2 - \underbrace{(dr^2 + r^2 d\Omega^2)}_{d\vec{x}^2} \quad \text{with} \quad d\Omega^2 := d\theta^2 + \sin^2\theta\, d\varphi^2 \tag{13.1}$$

⊲ Most general *spherically symmetric* metric (= invariant under spatial rotations $\vec{x}' = R\vec{x}$):

$$ds^2 = A(r,t)\,dt^2 - B(r,t)\,dr^2 + 2C(r,t)\,dt\,dr - \underbrace{D(r,t)}_{=r^2\ (wlog)}\, d\Omega^2 \tag{13.2}$$

A, B, C, D: Undetermined functions

- A metric that is spherically symmetric should allow for coordinates that reflect this symmetry. This means that the metric "looks the same" in all directions, i.e., no coefficient $g_{\mu\nu}$ can depend on θ or φ (above A, B, C, D). Furthermore, the metric should not contain any off-diagonals that mix angles $d\theta$ and $d\varphi$ with either time dt or the radial part dr. For fixed time t and radius r, a spherically symmetric metric must describe, well, a sphere, so that the only allowed length element is $d\Omega^2$, possibly scaled by a constant.

  These statement are sloppy. What we really want is a metric that has *three* (linearly independent) *space-like* ← *Killing vector fields* that satisfy the Lie algebra $\mathfrak{so}(3)$ (the algebra of angular momentum operators in quantum mechanics) – and therefore represent spatial rotations; such spacetimes are called ↑ *spherically symmetric* because their isometries (generated by Killing vectors) include the rotation group SO(3). One can then show that for such spherically symmetric spacetimes there exist coordinates in which the metric has the form (13.2). This is similar to Section 11.5 were we studied how the existence of a time-like Killing vector restricts the components of the metric in appropriately chosen coordinates.

- Note that we do not assume that the metric is ← *stationary* or even ← *static*, nor do we restrict its asymptotic behavior.

- That one can always choose coordinates where $D(r,t) = r^2$ is easy to see: Simply define new coordinates $\bar{r} := \sqrt{D(r,t)}$ and $\bar{t} := t$, and use the transformation $d\bar{r} = \partial_r\sqrt{D(r,t)}\,dr + \partial_t\sqrt{D(r,t)}\,dt$ and $d\bar{t} = dt$ to rewrite ds^2. This modifies the prefactors $A \to \bar{A}$, $B \to \bar{B}$ and $C \to \bar{C}$, but does not introduce additional terms beyond $d\bar{t}^2$, $d\bar{r}^2$ and $d\bar{t}d\bar{r}$ in the metric. Finally, rename $\bar{r} \mapsto r$, $\bar{t} \mapsto t$, $\bar{A} \mapsto A$ etc.

2 | Define new time coordinate $\bar{t} = \bar{t}(t,r)$ and a suitable function $\omega = \omega(t,r)$ such that

$$d\bar{t} = \omega(A dt + C dr) \tag{13.3}$$

That this is always possible is straightforward to see: First, note that the expression $Adt + Cdr$ is not necessarily an ↑ *exact* differential form, i.e., it is not guaranteed for $\bar{t}(t,r)$ to exist. This is why we need the additional function $\omega(t,r)$. On a suitable domain (it must be ↑ *contractible*), the ↑ *Poincaré lemma* tells us that every ↑ *closed form* is exact. This means that if we can choose $\omega(t,r)$ such that $\omega(Adt + Cdr)$ becomes *closed*, we know that $\bar{t}(t,r)$ exists such that Eq. (13.3) holds.

A differential form is closed if its exterior derivative vanishes:

$$0 \stackrel{!}{=} d[\omega(Adt + Cdr)] = [\partial_r(\omega A) - \partial_t(\omega C)]\, dr \wedge dt\,. \tag{13.4}$$

This condition is equivalent to a first-order partial differential equation for ω,

$$\partial_r(\omega A) = \partial_t(\omega C) \quad \Leftrightarrow \quad (\partial_r\omega)A + (\partial_r A)\omega = \partial_t(\omega)C + \partial_t(C)\omega\,, \tag{13.5}$$

for which you need to find only one non-zero solution ω, given the functions A and C. Such a solution is called ↑ *integrating factor*.

Eq. (13.2) $\xrightarrow{\text{Eq. (13.3)}}$

$$A\mathrm{d}t^2 + 2C\,\mathrm{d}t\,\mathrm{d}r \stackrel{\circ}{=} \frac{\mathrm{d}\bar{t}^2}{A\omega^2} - \frac{C^2\mathrm{d}r^2}{A} \tag{13.6}$$

This trick eliminates the mixed term $\mathrm{d}t\,\mathrm{d}r$ (we drop again all bars and rename prefactors):

$$\mathrm{d}s^2 = A(r,t)\,\mathrm{d}t^2 - B(r,t)\,\mathrm{d}r^2 - r^2\,\mathrm{d}\Omega^2 \tag{13.7}$$

3 | Lorentz signature of $\mathrm{d}s^2 \to A > 0$ and $B > 0 \to$ Define $A \equiv e^\nu c^2$ and $B \equiv e^\lambda \to$

Here $\nu = \nu(r,t)$ and $\lambda = \lambda(r,t)$ are undetermined functions:

$$\mathrm{d}s^2 = e^\nu\,\mathrm{d}(ct)^2 - e^\lambda\,\mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right) \equiv g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \tag{13.8}$$

Note that our coordinates are $(x^0, x^1, x^2, x^3) = (ct, r, \theta, \varphi)$.

In these coordinates the only non-zero components of the metric tensor are:

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & & & \\ & g_{11} & & \\ & & g_{22} & \\ & & & g_{33} \end{pmatrix}_{\mu\nu} = \begin{pmatrix} e^\nu & & & \\ & -e^\lambda & & \\ & & -r^2 & \\ & & & -r^2\sin^2\theta \end{pmatrix}_{\mu\nu} \tag{13.9}$$

4 | Our final goal is to solve the Einstein field equations for a point mass using the rotation symmetric ansatz Eq. (13.8). To this end, we need the curvature tensor, and therefore the …

Christoffel symbols:

A straightforward but tedious calculation yields the non-zero components:

Eq. (13.9) $\xrightarrow{\text{Eq. (10.79)}}$

$$\Gamma^0{}_{00} = \frac{\dot\nu}{2}, \qquad \Gamma^0{}_{01} = \frac{\nu'}{2}, \qquad \Gamma^0{}_{11} = \frac{\dot\lambda}{2}e^{\lambda-\nu} \tag{13.10a}$$

$$\Gamma^1{}_{00} = \frac{\nu'}{2}e^{\nu-\lambda}, \quad \Gamma^1{}_{01} = \frac{\dot\lambda}{2}, \qquad \Gamma^1{}_{11} = \frac{\lambda'}{2} \tag{13.10b}$$

$$\Gamma^1{}_{22} = -re^{-\lambda}, \quad \Gamma^1{}_{33} = -re^{-\lambda}\sin^2\theta$$

$$\Gamma^2{}_{12} = \frac{1}{r}, \qquad \Gamma^2{}_{33} = -\sin\theta\cos\theta \tag{13.10c}$$

$$\Gamma^3{}_{13} = \frac{1}{r}, \qquad \Gamma^3{}_{23} = \cot\theta \tag{13.10d}$$

with abbreviations $\dot\Box \equiv \frac{\partial\Box}{\partial(ct)} = \frac{\partial\Box}{\partial x^0}$ and $\Box' \equiv \frac{\partial\Box}{\partial r} = \frac{\partial\Box}{\partial x^1}$.

We extend this convention to higher derivatives in the obvious way.

All not listed components either vanish or are given by the symmetry of the Christoffel symbols.

5 | Einstein field equations:

Another straightforward (but even more tedious) calculation yields the non-zero components of the Einstein tensor. With these, the Einstein field equation reads:

$$G_0{}^0 = e^{-\lambda}\left(\tfrac{1}{r^2} - \tfrac{\lambda'}{r}\right) - \tfrac{1}{r^2} = -\kappa T_0{}^0 \tag{13.11a}$$

$$G_0{}^1 = e^{-\lambda}\tfrac{\dot\lambda}{r} = -\kappa T_0{}^1 \tag{13.11b}$$

$$G_1{}^1 = e^{-\lambda}\left(\tfrac{1}{r^2} + \tfrac{\nu'}{r}\right) - \tfrac{1}{r^2} = -\kappa T_1{}^1 \tag{13.11c}$$

$$G_2{}^2 = \left\{\begin{array}{l} \tfrac{1}{2}e^{-\lambda}\left(\nu'' + \tfrac{\nu'^2}{2} + \tfrac{\nu'-\lambda'}{r} - \tfrac{\nu'\lambda'}{2}\right) \\ -\tfrac{1}{4}e^{-\nu}\left(2\ddot\lambda + \dot\lambda^2 - \dot\lambda\dot\nu\right)\end{array}\right\} = -\kappa T_2{}^2 \tag{13.11d}$$

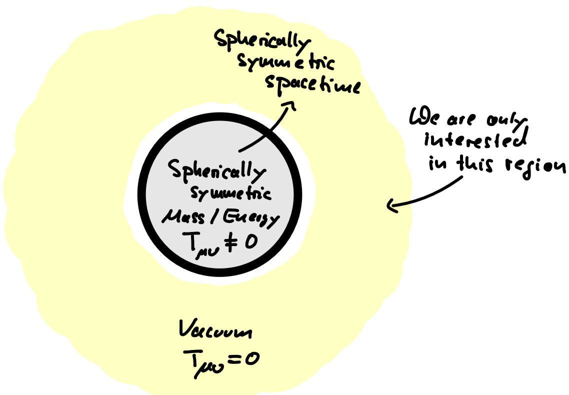
$$G_3{}^3 = G_2{}^2 = -\kappa T_3{}^3 \tag{13.11e}$$

All other components of the Einstein tensor vanish.

Our rotation symmetric ansatz Eq. (13.9) for the metric of course imposes restrictions on the form of the energy-momentum tensor for which solutions exists. Note that the Einstein tensor contains *second*-order derivatives for it derives from the curvature tensor.

13.1.2. Birkhoff's theorem

6 | $\sphericalangle$ Spherically symmetric solutions *in vacuum*: $T_{\mu\nu} = 0$



This means that we are interested in the metric *outside* of spherically symmetric bodies (like planets and stars). As this is exactly were we would like to test GENERAL RELATIVITY (e.g., by following test particles on their geodesics), this simplifications is actually well motivated.

7 | $\sphericalangle$ First three equations of Eq. (13.11):

$$\text{Eq. (13.11a)} \quad \Leftrightarrow \quad e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} = 0 \tag{13.12a}$$

$$\text{Eq. (13.11b)} \quad \Leftrightarrow \quad \dot\lambda = 0 \tag{13.12b}$$

$$\text{Eq. (13.11c)} \quad \Leftrightarrow \quad e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = 0 \tag{13.12c}$$

Eq. (13.12b) $\rightarrow \lambda = \lambda(r)$

8 | Eq. (13.12a) + Eq. (13.12c) →

$$\lambda'(r) + \nu'(r,t) = 0 \quad \Leftrightarrow \quad \nu'(t,r) = -\lambda'(r) \tag{13.13}$$

Integration yields the form

$$\nu(t,r) = \nu(r) + f(t)\,. \tag{13.14}$$

The fact that $\nu(r)$ relates to $\lambda(r)$ is not important right now. The crucial point is that the space and time dependency of $\nu(t,r)$ separated into two summands:

$\sphericalangle$ Coordinate transformation $\bar{t} = \bar{t}(t)$ with $\mathrm{d}\bar{t} = e^{f/2}\mathrm{d}t$:

$$e^{\nu(t,r)}\mathrm{d}(ct)^2 \overset{13.14}{=} e^{\nu(r)}e^{f(t)}\mathrm{d}(ct)^2 = e^{\nu(r)}\,\mathrm{d}(c\bar{t})^2 \tag{13.15}$$

That such a coordinate transformation always exists is easy to see: The integral $\bar{t}(t) := \int_{t_0}^{t} e^{f(s)/2}\mathrm{d}s$ does the job by construction.

Eq. (13.8) $\xrightarrow{\bar{t}\mapsto t}$ Most general spherically symmetric solution in vacuum:

$$\mathrm{d}s^2 = e^{\nu(r)}\,\mathrm{d}(ct)^2 - e^{\lambda(r)}\,\mathrm{d}r^2 - r^2\,\mathrm{d}\Omega^2 \tag{13.16}$$

Our new insight is that $\lambda = \lambda(r)$ and $\nu = \nu(r)$ do not depend on the time coordinate:

→ Metric is ← *static*.

9 | Eq. (13.13) is still valid: [combine Eqs. (13.13) and (13.14)]

$$\lambda'(r) + \nu'(r) = 0 \quad \Rightarrow \quad \lambda(r) + \nu(r) = 0 \tag{13.17}$$

There is of course also an integration constant. But this constant can be absorbed in the term $e^{\nu(r)}\mathrm{d}(ct)^2$ by another coordinate transformation (rescaling) of the time coordinate.

10 | Let us once again go back to the Einstein field equations:

$$\text{Eq. (13.12a)} \quad \Leftrightarrow \quad e^{-\lambda(r)}\left(1 - \lambda' r\right) = 1 \tag{13.18}$$

Use substitution $\alpha(r) := e^{-\lambda(r)}$ →

$$\alpha + \alpha' r = 1 \tag{13.19}$$

→ Solution:

$$\alpha = 1 + \frac{a}{r} = e^{-\lambda} \overset{13.17}{=} e^{\nu} \quad \text{with integration constant } a. \tag{13.20}$$

$\sphericalangle$ Spatial infinity $r\to\infty$: [Use $\lim_{r\to\infty} e^{\lambda} = 1 = \lim_{r\to\infty} e^{\nu}$.]

$$\lim_{r\to\infty} \mathrm{d}s^2 \overset{13.16}{=} \mathrm{d}(ct)^2 - \mathrm{d}r^2 - r^2\,\mathrm{d}\Omega^2 = \langle\text{Minkowski space}\rangle \tag{13.21}$$

→ Metric is *asymptotically flat*.

11 | Check that …

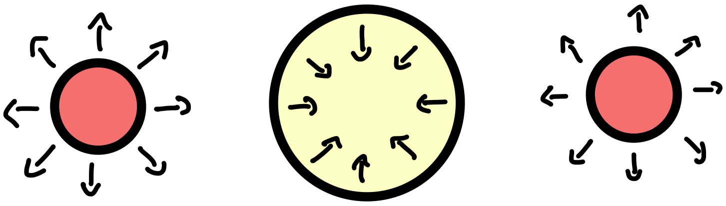
- Eqs. (13.11a) and (13.17) → Eq. (13.11c) solved ✓
- Eq. (13.20) $\overset{\circ}{\to}$ Eqs. (13.11d) and (13.11e) solved ✓

12 | ⁂ *Birkhoff's theorem*:

We can summarize our results as follows:

> Every *spherically symmetric* solution of the Einstein field equations *in vacuum* is *static* and *asymptotically flat*.

- The theorem was proven by American mathematician GEORGE DAVID BIRKHOFF in 1923 [203]. However, the same result was obtained already in 1921 by Norwegian physicist JØRG TOFTE JEBSEN [204]. Birkhoff's theorem is therefore a typical example for ↑ *Stigler's law* [205] according to which no scientific discovery is named after its original discoverer.
- If you think about it, this result is quite surprising as we didn't exploit any properties of the energy-momentum tensor that produces the gravitational field except for its spherical symmetry. This means that our result holds also for *time-dependent* distributions of mass/energy – as long as the time dependence does not break the spherical symmetry. For example, consider a pulsating (non-rotating) star:



Birkhoff's theorem demands that the metric outside of this star is nonetheless *static* and asymptotically flat. This implies in particular that such a time-dependent object *cannot emit gravitational waves*!

[A similar situation occurs when a dying star explodes in a supernova: If the explosion is spherically symmetric, such an event cannot emit gravitational waves.]

13.1.3. The Schwarzschild metric

13 | The derivation above yields the most general solution of the vacuum EFEs that are spherically symmetric:

Eqs. (13.16) and (13.20) →

$$ds^2 = \left(1+\frac{a}{r}\right)\mathrm{d}(ct)^2 - \left(1+\frac{a}{r}\right)^{-1}\mathrm{d}r^2 - r^2\,\mathrm{d}\Omega^2 \tag{13.22}$$

→ The parameter a must be determined by the mass of the object that generates this metric.

14 | ⊲ Correspondence principle:

In the non-relativistic weak-field limit, we must recover the Newtonian gravitational potential:

$$1+\frac{a}{r} \overset{13.22}{=} g_{00} \overset{11.65}{\approx} 1+\frac{2\phi}{c^2} = 1-\frac{2GM/c^2}{r} \quad \text{with} \quad \phi = -\frac{GM}{r} \tag{13.23}$$

Here M is the mass of the central spherically symmetric body.

$\to a = -\frac{2GM}{c^2} \equiv -r_s$ with the …

$$r_s = \frac{2GM}{c^2} \qquad \text{⁂ } \textit{Schwarzschild radius} \tag{13.24}$$

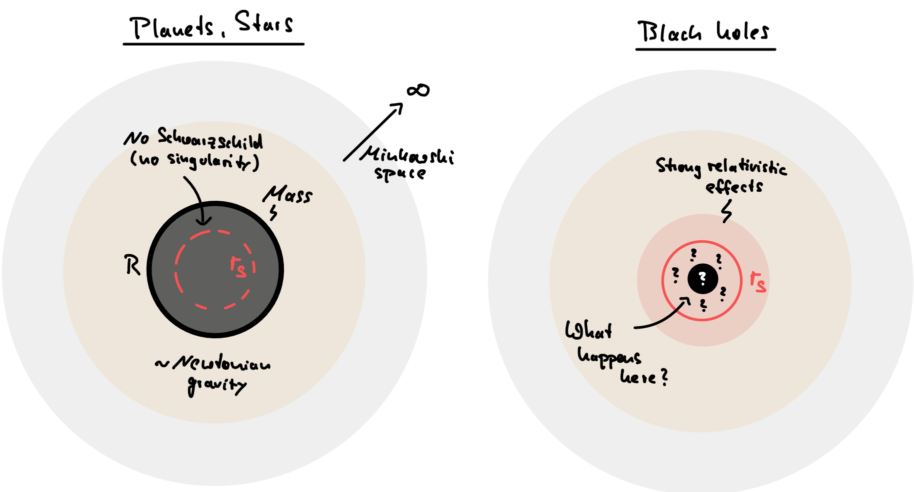
Thus we finally find the ⁂ *Schwarzschild metric*

$$ds^2 = \left(1 - \frac{r_s}{r}\right) d(ct)^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2 d\Omega^2 \tag{13.25}$$

expressed in ⁂ *Schwarzschild coordinates* (ct, r, θ, φ) with $d\Omega^2 = d\theta^2 + \sin^2\theta \, d\varphi^2$.

15 | Comments:

- ¡! Do not make the mistake to interpret t and r as measurable times and distances, respectively. These are just *coordinates* and one must compute coordinate independent proper times and distances to check if, how, and where they relate to observable quantities (→ *below*). (Recall the remarks in Section 9.2 about the role played by coordinates in GENERAL RELATIVITY.) Note also that for $r < r_s$ the coordinate t is actually *space-like* whereas r is time-like.
- In Schwarzschild coordinates, the metric Eq. (13.25) has *two* singularities:
  1. On the sphere with $r = r_s$ (→ *event horizon*) the prefactor of dr^2 diverges and the prefactor of dt^2 vanishes. You show on ➔ Problemset 6 that this singularity is an artifact of the Schwarzschild *coordinates*, and that it can be remedied by choosing better coordinates (e.g. ↑ *Kruskal–Szekeres coordinates*). The *metric* can then be smoothly extended beyond the horizon without anything fancy happening on the horizon itself.
  2. At $r = 0$ the prefactor of dt^2 blows up and the prefactor of dr^2 vanishes. In contrast to the coordinate singularity at $r = r_s$, the singularity at $r = 0$ of the interior solution is "physical" in the sense that there coordinate-*independent* quantities (scalars built from the curvature tensor) diverge. However, keep in mind that for "normal" bodies like planets and stars, the Schwarzschild metric is not valid in the interior anyway, so that this singularity has no physical relevance in these scenarios. Only for black holes this singularity is relevant as it heralds the breakdown of GENERAL RELATIVITY.
- Inspection of Eq. (13.25) shows that the ratio r_s/r quantifies the deviations from flat Minkowski space. Because the Schwarzschild solution Eq. (13.25) is only valid *outside* of the mass, relativistic effects become important if one can approach the body to $r \sim r_s$, i.e., the (coordinate) radius R of the body must be of the order of the Schwarzschild radius. Conversely, for bodies with $R \gg r_s$ it is necessarily $r \gg r_s$ such that the dominant effect of the Schwarzschild metric is described by Newtonian gravity:

  The situation of a → *black hole* where $r < r_s$ is possible will be discussed in Section 13.3; in the following we assume $r \geq R > r_s$ so that neither the coordinate singularity at $r = r_s$ nor the physical singularity at $r = 0$ are relevant.

- ¡! Do not forget that the Schwarzschild solution Eq. (13.25) is only valid in vacuo, i.e., *outside* the gravitating mass. The metric in the *interior* is different from Eq. (13.25) (in particular: non-singular). This means that if the Schwarzschild radius $r_s < R$ is "buried" in the body, *it has no physical significance.* There is no event horizon close to the center of Earth!

  For the exact solution of the interior of a star see ↑ WEINBERG [120] (§11.1, pp. 299–304).

- Because of the coordinate singularity at $r = r_s$, the Schwarzschild solution in Eq. (13.25) actually separates into *two* independent solutions for the EFEs in vacuum: The extended *outer* solution for $r > r_s$ (with time-like coordinate t) and the bounded *inner* solution for $0 < r < r_s$ (with time-like coordinate r). Since the metric is undefined at $r = r_s$, it is a priori unclear whether (and if, how) these two "patches" can be glued together to form a single, contiguous spacetime that solves the vacuum EFEs. That (and how) this is possible can be seen for example in ↑ *Kruskal–Szekeres coordinates* (➔ Problemset 6).

- If one plugs in the numbers, the Schwarzschild radius of a spherical mass M is roughly

$$r_s \sim 3 \times \left(\frac{M}{M_\odot}\right) \text{km} \tag{13.26}$$

  where $M_\odot$ is the mass of the Sun. One finds for example:

	r_s [m]	r_s/R
Erde	9×10^{-3}	10^{-9}
Sonne	3×10^{3}	10^{-6}
White dwarf	3×10^{3}	3×10^{-4}
Neutron star	3×10^{3}	0.3

  This explains why the Newtonian approximation has been so successful in our Solar System.

  It is clear that we should compare r_s to the *coordinate* radius R of the spherical body, i.e., the radial Schwarzschild coordinate $r = R$ where the surface of the body is located (since the terms r_s/r compare r_s to the *coordinate* r). Remembering our warning above (that coordinates cannot directly be identified with physical quantities), you might object that equating R with the *measured* radius of (e.g.) Earth is not justified. This is indeed a valid objection; however, → *below* we will see that the coordinate radius has a straightforward physical meaning – which justifies the numbers above (although the interpretation is not the one you might expect).

- According to Birkhoff's theorem, the Schwarzschild metric is the *unique* solution of the vacuum field equations outside of a *spherically symmetric, non-rotating, uncharged mass*. That the mass is non-rotating is important, because a finite angular momentum breaks the rotation symmetry of the problem. That the mass is uncharged is important, because otherwise the electromagnetic field *outside* the mass would be non-zero and our assumption $T_{\mu\nu} = 0$ would be invalid.

- One can loosen these restrictions and solve the EFEs for more general scenarios:

Rotating?	**Charged?**	**Metric**	**Ref.**	**Found**
✗	✗	← *Schwarzschild*	[201]	1916
✗	✓	↑ *Reisser-Nordström*	[206–209]	1916
✓	✗	↑ *Kerr*	[210]	1963
✓	✓	↑ *Kerr-Newman*	[211, 212]	1965

  Because most celestial bodies rotate, these generalizations (in particular the Kerr metric) are often more useful to describe real phenomena than the Schwarzschild metric (like black holes). However, for slowly rotating bodies the Schwarzschild metric often provides good approximations to explain a variety of phenomena (→ Section 13.2; but not always, → *Lense-Thirring effect* in ➔ Problemset 6).

- In his derivation, Schwarzschild used both *time-independence* and *asymptotic flatness* as independent assumptions [201]. The contribution by Birkhoff and Jebsen was to show that both assumptions are superfluous [203, 204]: That the solution must be static and asymptotically flat is already implied by its rotational symmetry.

16 | Proper time:

Let us now study how the Schwarzschild coordinates relate to measurable proper time:

i | ⊲ Ideal clock at rest in Schwarzschild coordinates:

→ Proper time:

$$\mathrm{d}\tau \overset{11.10}{=} \frac{1}{c}\mathrm{d}s \overset{13.25}{=} \sqrt{1-\frac{r_s}{r}}\,\mathrm{d}t \tag{13.27}$$

→ $\Delta\tau < \Delta t$ for $r_s < r < \infty$

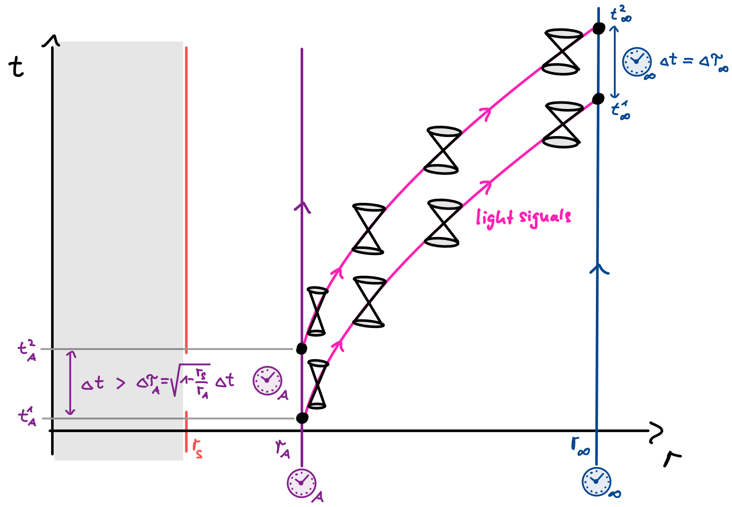
ii | ⊲ Asymptotic observer at $r \to \infty$:

$$\lim_{r\to\infty} \mathrm{d}\tau = \mathrm{d}t \tag{13.28}$$

→ We can conclude:

$$\boxed{\text{Schwarzschild time } t = \text{Proper time of observer at spatial infinity}} \tag{13.29}$$

iii | In summary, the clocks of stationary observers at finite distance to the mass run always *slower* than the clocks at spatial infinity. The closer the clock to the Schwarzschild radius, the slower it ticks. We can illustrate this as follows:



- To draw the null cones in a Schwarzschild rt-diagram, note that $\mathrm{d}s^2 \overset{!}{=} 0$ implies

$$\frac{\mathrm{d}(ct)}{\mathrm{d}r} = \pm\left(1-\frac{r_s}{r}\right)^{-1} \tag{13.30}$$

for constant θ and φ. So for $r \to \infty$ the cones open with 90°, as in flat Minkowski space; for $r \to r_s$ the cones close up and become degenerate at the Schwarzschild radius.

17 | Proper distance:

How do the Schwarzschild coordinates relate to proper distances?

i | $\sphericalangle$ Time slice $t = \text{const}$ ($\mathrm{d}t = 0$)

$\rightarrow$ Spatial metric: [For a formal definition see Eq. (11.30) or Eq. (11.27).]

$$\mathrm{d}l^2 = \left(1 - \frac{r_s}{r}\right)^{-1} \mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\varphi^2\right) \tag{13.31}$$

ii | $\sphericalangle$ Circumference of a great circle $\mathcal{C}$ of *coordinate* radius r ($\theta = \frac{\pi}{2}$):

$$L[\mathcal{C}] := \underbrace{\int_{\mathcal{C}} \mathrm{d}l}_{\text{Coordinate independent}} = \underbrace{r \int_0^{2\pi} \mathrm{d}\varphi}_{\text{Coordinate dependent}} = 2\pi r \tag{13.32}$$

Similarly, one finds $A[\mathcal{S}] = 4\pi r^2$ for the *surface* of a sphere $\mathcal{S}$ with coordinate radius r.

$\rightarrow$ The coordinate r directly relates to lengths of circles (and areas of spheres).

Note that both $L[\mathcal{C}]$ and $A[\mathcal{S}]$ are geometric (= coordinate independent) quantities.

iii | But what about *radial* proper distances?

$\sphericalangle$ Radial segment $\mathcal{L}$ from r_1 to r_2 ($\theta = \text{const}$ and $\varphi = \text{const}$):

$$L[\mathcal{L}] := \int_{\mathcal{L}} \mathrm{d}l = \int_{r_1}^{r_2} \frac{\mathrm{d}r}{\sqrt{1 - \frac{r_s}{r}}} =: \Delta R(r_1, r_2) > r_2 - r_1 \tag{13.33}$$

Note that we cannot compute distances from the center $r = 0$ because, first, we would integrate over the coordinate singularity (and start at the singularity at $r = 0$), and second, for $r < r_s$ the coordinate becomes time-like and the integral actually measures a *time* and not a length! This is why we consider distances between two points with radial coordinates $r_2, r_1 > r_s$.

We conclude:

$$\boxed{\text{The radial proper distance is } \textit{larger} \text{ than the coordinate distance.}} \tag{13.34}$$

iv | $\sphericalangle$ Two great circles $\mathcal{C}_i$ with radii $r_2 > r_1 \rightarrow$

$$\frac{\delta U}{\delta R} := \frac{L[\mathcal{C}_2] - L[\mathcal{C}_1]}{\Delta R(r_1, r_2)} = \frac{2\pi(r_2 - r_1)}{\Delta R(r_1, r_2)} \overset{13.33}{<} \frac{2\pi(r_2 - r_1)}{r_2 - r_1} = 2\pi \tag{13.35}$$

This means that the circumference varies "less than usual": $\delta U < 2\pi\delta R$.

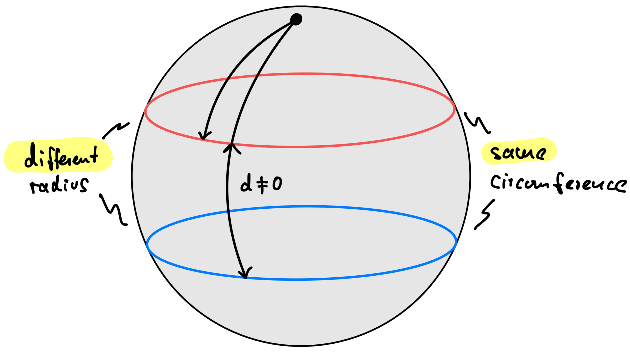
Compare this to Euclidean geometry:

$$\frac{\delta U}{\delta R} := \frac{2\pi r_2 - 2\pi r_1}{r_2 - r_1} = 2\pi \quad \Rightarrow \quad \delta U = 2\pi \delta R \tag{13.36}$$

Note that the ratio defined in Eq. (13.35) makes use of geometric properties of the space(time) only; i.e., both $L[\mathcal{C}_i]$ and Δd are (in principle) measurable quantities that do not depend on coordinates.

→ Space is *non-Euclidean*!

The fact that this ratio is *smaller* than 2π tells us that the spatial curvature is *positive*. For example, a two-dimensional sphere has positive curvature and the same feature:



(This is an extreme example where the ratio is zero.)

v | Let us approximate the measure Eq. (13.35) and apply it to the Solar System to get a feeling for how non-Euclidean space actually is in our neighborhood:

We assume $r_s \ll r$ and $r_2 - r_1 \gg r_s$ (which is satisfied for all situations in the Solar System).

Eq. (13.33) →

$$\Delta R(r_1, r_2) \approx \int_{r_1}^{r_2} \mathrm{d}r \left(1 + \frac{1}{2}\frac{r_s}{r}\right) = r_2 - r_1 + \frac{r_s}{2} \ln \frac{r_2}{r_1} \tag{13.37}$$

With this and Eq. (13.35) we find:

$$\frac{2\pi(r_2 - r_1)}{\Delta R(r_1, r_2)} \overset{\circ}{\approx} 2\pi\Big[1 - \underbrace{\frac{1}{2}\left(\frac{r_s}{r_2 - r_1}\right) \ln \frac{r_2}{r_1}}_{\text{Non-Euclid. correction } \epsilon}\Big]. \tag{13.38}$$

For example, let $r_1 = 7 \times 10^8$ m be the radius of the Sun and $r_2 = 5.8 \times 10^{10}$ m the semi-major axis of Mercury. With the Schwarzschild radius $r_s = 3 \times 10^3$ m (of the Sun) one finds the non-Euclidean correction $\epsilon \approx 10^{-7}$.

→ The deviations from Euclidean geometry in the Solar System are miniscule.

This explains why the Euclidean space used in Newtonian mechanics is such a good approximation to describe the Solar System!

18 | Alternative coordinates:

There is a zoo of different coordinate systems adapted to the Schwarzschild metric, all with distinct advantages and disadvantages. Here we introduce one alternative coordinate system to demonstrate that the singularity at $r = r_s$ is an artifact of Schwarzschild coordinates:

For a motivation of the widely used ↑ *Kruskal–Szekeres coordinates*: ➔ Problemset 6.

i | ⊲ Coordinate transformation $\bar{r} = \bar{r}(r)$ with

$$r \equiv \left(1 + \frac{r_s}{4\bar{r}}\right)^2 \bar{r} \tag{13.39}$$

and $r \geq r_s \;\Leftrightarrow\; \bar{r} \geq r_s/4$.

Eq. (13.25) $\overset{\circ}{\rightarrow}$

$$\mathrm{d}s^2 = \left(\frac{1 - \frac{r_s}{4\bar{r}}}{1 + \frac{r_s}{4\bar{r}}}\right)^2 \mathrm{d}(ct)^2 - \left(1 + \frac{r_s}{4\bar{r}}\right)^4 \underbrace{(\mathrm{d}\bar{r}^2 + \bar{r}^2\mathrm{d}\Omega^2)}_{\equiv \mathrm{d}\bar{x}^2 + \mathrm{d}\bar{y}^2 + \mathrm{d}\bar{z}^2} \tag{13.40}$$

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with ⁂ *isotropic coordinates* $(c\bar{t}, \bar{x}, \bar{y}, \bar{z})$ or $(c\bar{t}, \bar{r}, \theta, \varphi)$.

Note that the $\bar{r}$-dependent scaling now affects all spatial coordinates equally; hence *isotropic* coordinates [cf. Eq. (13.25)].

ii | *Important:* No divergence/singularity for $\bar{r} \to r_s/4$ in Eq. (13.40) !

Although the *divergence* (singularity) at the event horizon is gone, the metric is still *degenerate* at $\bar{r} = r_s/4$ since the component $\bar{g}_{00} = 0$ vanishes [← Eq. (3.46)]. The ↑ *Kruskal–Szekeres coordinates* you study in ➔ Problemset 6 do not have this problem and are non-degenerate and non-singular on the event horizon.

iii | ◁ Weak field limit $\bar{r} \gg r_s \overset{\circ}{\to}$ (expand linearly in $\frac{r_s}{\bar{r}}$)

$$\mathrm{d}s^2 \approx \left(1 - \frac{r_s}{\bar{r}}\right) \mathrm{d}(ct)^2 - \left(1 + \frac{r_s}{\bar{r}}\right) \left(\mathrm{d}\bar{r}^2 + \bar{r}^2 \mathrm{d}\Omega^2\right) \tag{13.41}$$

19 | Cosmological constant:

Retracing the solution in Section 13.1.2 – but now including the cosmological constant in the EFEs – yields the ⁂ *Schwarzschild de Sitter metric*

$$\mathrm{d}s^2 = \left(1 - \frac{r_s}{r} - \frac{\Lambda r^2}{3}\right) \mathrm{d}(ct)^2 - \left(1 - \frac{r_s}{r} - \frac{\Lambda r^2}{3}\right)^{-1} \mathrm{d}r^2 - r^2 \mathrm{d}\Omega^2 \,. \tag{13.42}$$

[↑ *de Sitter space* is the maximally symmetric (= 10 Killing vectors) spacetime with constant *positive* scalar curvature ($R > 0$); you can think of it as the generalization of spheres in Euclidean space. De Sitter space is the maximally symmetric vacuum solution of the EFEs with positive cosmological constant – analog to Minkowski space for the case of vanishing cosmological constant.]

Due to the additional terms in Eq. (13.42), the asymptotic metric for $r \to \infty$ is no longer flat Minkowski space but positively curved de Sitter space. In the non-relativistic limit, the gravitational potential can be identified via Eq. (11.65) as

$$\phi = -\frac{GM}{r} - \frac{c^2 \Lambda}{6} r^2 \,. \tag{13.43}$$

This is a modification of Newtonian gravity and consistent with our previous result Eq. (12.42).