↓Lecture 24 [11.06.24]

11.4. The Hilbert Energy-Momentum Tensor

We have now discussed two covariant generalizations of classical theories that are valid on arbitrary spacetimes, specified by a given Riemannian metric $g_{\mu\nu}$. In this chapter, we study the implications of the general covariance of such theories to understand under which circumstances they feature conserved quantities. As a bonus, the central results of this chapter will be crucial for the derivation of the Einstein field equations in the next Chapter 12.

1 | ⊲ Generally covariant theory describing *matter fields* ϕ:

$$S_g[\phi] = \int \underbrace{\mathrm{d}^4x\sqrt{g}}_{\text{Scalar}}\,\underbrace{L(\phi,\partial\phi,g,\partial g)}_{\text{Scalar}} \equiv \int \mathrm{d}^4x\,\underbrace{\mathcal{L}(\phi,\partial\phi,g,\partial g)}_{\text{Scalar } density} \tag{11.84}$$

Recall Section 3.4 for the definition of ← *tensor densities*.

- The following is valid for arbitrary families of matter fields $\{\phi_k\}$; we omit the index k. In particular, we do not assume that the fields transform as scalars, they can be arbitrary tensor fields. The only important thing is that they are combined appropriately to a Lagrangian L that transforms as a scalar.
- "Matter" here refers to all degrees of freedom that are *not* the metric $g_{\mu\nu}$. So one example for ϕ would be the gauge field A^μ of classical electrodynamics, discussed in Section 11.3 with the action Eq. (11.78).
- We use the subscript g do indicate that the action depends on the metric (e.g., through covariant derivatives). If we consider these theories "stand alone", i.e., on a fixed spacetime background, the metric plays the role of a parameter (not a dynamical field), which motivates the subscript notation.
- The expression in Eq. (11.84) actually requires an additional prefactor $\frac{1}{c}$ for dimensional reasons because we measure time coordinates in units of length ($x^0 = ct$); we omit the prefactor because it is irrelevant in the following and would cancel anyway.

2 | Diffeomorphism invariance:

⊲ Arbitrary coordinate transformation $\bar{x} = \varphi(x) \Leftrightarrow x = \varphi^{-1}(\bar{x}) \rightarrow$

$$\bar{\phi}(\bar{x}) := \mathcal{F}_\phi(\phi(x)) \quad \Leftrightarrow \quad \bar{\phi}(x) := \mathcal{F}_\phi(\phi(\varphi^{-1}(x))) \tag{11.85a}$$

$$\bar{g}(\bar{x}) := \mathcal{F}_g(g(x)) \quad \Leftrightarrow \quad \bar{g}(x) := \mathcal{F}_g(g(\varphi^{-1}(x))) \tag{11.85b}$$

$\mathcal{F}$ is shorthand for the transformation of the field components (e.g., $\mathcal{F} = \mathbb{1}$ for a scalar).

(Note that in x is a dummy variable in the right column; you can call it whatever you like.)

For example, the metric tensor transforms as

$$\bar{g}(\bar{x}) = \mathcal{F}_g(g(x)) \quad \Leftrightarrow \quad \bar{g}_{\mu\nu}(\bar{x}) = \frac{\partial x^\alpha}{\partial\bar{x}^\mu}\frac{\partial x^\beta}{\partial\bar{x}^\nu}g_{\alpha\beta}(x)\,. \tag{11.86}$$

By definition, S_g describes a generally covariant theory iff

$$L(\bar\phi(\bar x),\bar\partial\bar\phi(\bar x),\bar g(\bar x),\bar\partial\bar g(\bar x)) \overset{11.85}{=} L(\phi(x),\partial\phi(x),g(x),\partial g(x))\,. \tag{11.87}$$

This is a non-trivial constraint on the functional form of L that can be satisfied by constructing it from proper tensorial expressions to form a scalar.

For example, the Lagrangian of Maxwell theory (in vacuum) Eq. (11.78) satisfies

$$L(\bar A_\mu(\bar x),\bar\partial_\nu\bar A_\mu(\bar x),\bar g^{\mu\nu}(\bar x)) = -\frac{1}{16\pi}\bar g^{\mu\alpha}(\bar x)\bar g^{\nu\beta}(\bar x)\bar F_{\mu\nu}(\bar x)\bar F_{\alpha\beta}(\bar x) \tag{11.88a}$$

Use tensor transformation laws.

$$= -\frac{1}{16\pi}g^{\mu\alpha}(x)g^{\nu\beta}(x)F_{\mu\nu}(x)F_{\alpha\beta}(x) \tag{11.88b}$$

$$= L(A_\mu(x),\partial_\nu A_\mu(x),g^{\mu\nu}(x)) \tag{11.88c}$$

with $\bar F_{\mu\nu}(\bar x)=\bar\partial_\mu\bar A_\nu(\bar x)-\bar\partial_\nu\bar A_\mu(\bar x)$.

$\overset{\circ}{\rightarrow}$

$$S_{\bar g}[\bar\phi] \overset{\text{def}}{=} \int \mathrm{d}^4x\,\sqrt{\bar g(x)}L(\bar\phi(x),\partial\bar\phi(x),\bar g(x),\partial\bar g(x)) \tag{11.89a}$$

Rename dummy variables from x to $\bar x$ (no substitution!).

$$= \int \mathrm{d}^4\bar x\,\sqrt{\bar g(\bar x)}L(\bar\phi(\bar x),\bar\partial\bar\phi(\bar x),\bar g(\bar x),\bar\partial\bar g(\bar x)) \tag{11.89b}$$

Variable substitution: $\bar x=\varphi(x)$

$$\overset{11.87}{\underset{10.101}{=}} \int \mathrm{d}^4x\,\sqrt{g(x)}L(\phi(x),\partial\phi(x),g(x),\partial g(x)) \tag{11.89c}$$

$$\overset{\text{def}}{=} S_g[\phi] \tag{11.89d}$$

→ S_g is ⁂ *diffeomorphism invariant*

- ¡! This means that any generally covariant theory has a *symmetry* in that the value of the action functional does not change under the substitution of fields $(g,\phi)\mapsto(\bar g,\bar\phi)$ defined in Eq. (11.85) for arbitrary diffeomorphisms φ. Note that we must replace *both* the metric g and the matter fields ϕ for this to work despite the fact that g is not (yet) a dynamical field.
- What happened above is similar to what we did in Section 1.2, where we rephrased the invariance under Galilei transformations as an active symmetry of a dynamical equation.

  So far, we interpreted the map $\bar x=\varphi(x)$ as a *passive* coordinate transformation, i.e., both x and $\bar x$ are thought to describe *the same point* on the manifold. In this reading, the fields $\phi(x)$ and $\bar\phi(\bar x)$ describe the *same* physical state in different frames of reference. However, we can also interpret $\bar x=\varphi(x)$ as an *active* transformation in that φ actively moves the point corresponding to x to a *new* point corresponding to $\bar x$ (in the same coordinates!). In this interpretation, one calls φ a ↑ *diffeomorphism* (here in a particular chart), and we interpret $\bar\phi$ as a *new* function, defined on the *same* coordinates, describing a *different* state of the system.

3 | Diffeomorphism invariance is a *continuous* symmetry:

⊲ *Infinitesimal* diffeomorphisms:

$$\bar x^\mu_\varepsilon = \varphi_\varepsilon(x^\mu) = x^\mu + \delta_\varepsilon x^\mu = x^\mu + \varepsilon^\mu(x) \quad\text{with}\quad |\varepsilon^\mu| \ll 1 \tag{11.90}$$

You should think of the vector field $\varepsilon^\mu\partial_\mu \in TM$ generating the infinitesimal diffeomorphism:

$$\bar{\phi}_\varepsilon(x) \overset{11.85}{=} \phi(\varphi_\varepsilon^{-1}(x)) \overset{11.90}{=} \phi(x-\varepsilon) \overset{\text{Taylor}}{=} \phi(x) - \varepsilon^\mu(x)\partial_\mu\phi(x) + \mathcal{O}(\varepsilon^2) \tag{11.91}$$

(here for a scalar field) so that

$$\delta_\varepsilon\phi(x) \equiv \bar{\phi}_\varepsilon(x) - \phi(x) = -\varepsilon^\mu\partial_\mu\phi(x) \tag{11.92}$$

is the infinitesimal variation of the field (at the same point) due to the diffeomorphism; ← Eq. (6.79).

Remember ↓ *Noether's (first) theorem*: (← Eqs. (6.84) and (6.85))

Global continuous symmetry → Conserved current

What are the consequences of diffeomorphism invariance?

Note that there are two peculiarities that prevent us from applying Noether's (first) theorem to diffeomorphism invariance:

- Diffeomorphism invariance is a *local* symmetry.

  (Since the transformation $\varepsilon^\mu(x)$ can depend on the spacetime point x, one can consider transformations where $\varepsilon^\mu(x) = 0$ everywhere except for a compact subset of the manifold.)
- For the action to be invariant, we must also replace the metric $g \mapsto \bar{g}$ (which is not a dynamical field but a parameter).

4 | Thus let us proceed carefully and step by step:

i | ⊲ $\bar{\phi}_\varepsilon(x)$ and $\bar{g}_\varepsilon(x)$ defined by Eqs. (11.85) and (11.90) →

$$\delta_\varepsilon S := S_{\bar{g}_\varepsilon}[\bar{\phi}_\varepsilon] - S_g[\phi] \overset{11.89}{=} 0 \tag{11.93a}$$

$$= \int \mathrm{d}^4x \left[\mathcal{L}(\bar{\phi}_\varepsilon, \partial\bar{\phi}_\varepsilon, \bar{g}_\varepsilon, \partial\bar{g}_\varepsilon) - \mathcal{L}(\phi, \partial\phi, g, \partial g)\right] \tag{11.93b}$$

ii | We can split the variation into two parts in lowest order of ε:

$$\delta_\varepsilon S = \underbrace{S_{\bar{g}_\varepsilon}[\phi] - S_g[\phi]}_{=:\delta_g S} + \underbrace{S_g[\bar{\phi}_\varepsilon] - S_g[\phi]}_{=:\delta_\phi S} + \mathcal{O}\left(\varepsilon^2\right) = 0 \tag{11.94}$$

iii | ⊲ *Solutions* ϕ of the equations of motion ⇔

$$\forall\varepsilon^\mu(x): \quad \delta_\phi S = S_g[\bar{\phi}_\varepsilon] - S_g[\phi] = 0 \tag{11.95}$$

So that →

$$0 = \delta_\varepsilon S \doteq S_{\bar{g}_\varepsilon}[\phi] - S_g[\phi] \tag{11.96a}$$

$$= \int \mathrm{d}^4x \left[\mathcal{L}(\phi, \partial\phi, \bar{g}_\varepsilon, \partial\bar{g}_\varepsilon) - \mathcal{L}(\phi, \partial\phi, g, \partial g)\right] \tag{11.96b}$$

Here "$\doteq$" indicates an equality that is only valid "on shell", i.e., when the matter fields satisfy the matter equations of motion; for arbitrary fields, there are additional terms to be added. One can also say that "the equation is valid modulo EOMs".

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iv | Let us write the variation of the metric as follows:

$$\bar{g}_\varepsilon(x) \equiv g(x) + \delta_\varepsilon g(x) \quad \text{with} \quad |\delta_\varepsilon g(x)| \in \mathcal{O}(\varepsilon) \tag{11.97}$$

We derive an explicit expression for $\delta_\varepsilon g(x)$ → *below*.

Eq. (11.96b) →

$$\mathcal{L}(\phi, \partial\phi, \bar{g}_\varepsilon, \partial\bar{g}_\varepsilon) - \mathcal{L}(\phi, \partial\phi, g, \partial g) \overset{\mathcal{O}(\varepsilon)}{=} \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} \delta_\varepsilon g^{\mu\nu} + \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}_{,\lambda}} \delta_\varepsilon g^{\mu\nu}_{,\lambda} \tag{11.98a}$$

$$= \left[\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - \partial_\lambda \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}_{,\lambda}}\right] \delta_\varepsilon g^{\mu\nu} + \partial_\lambda \left(\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}_{,\lambda}} \delta_\varepsilon g^{\mu\nu}\right) \tag{11.98b}$$

Here we are sloppy and write $g^{\mu\nu}_{,\lambda} \equiv g^{\mu\nu}{}_{,\lambda}$ to streamline the notation. In the second step we used $\partial_\lambda(\delta_\varepsilon g^{\mu\nu}) = \delta_\varepsilon g^{\mu\nu}_{,\lambda}$.

v | ∢ *Compact* variation $\varepsilon^\mu(x) \to \delta_\varepsilon g^{\mu\nu} = 0$ on boundary of spacetime:

$$\text{Eqs. (11.96b) and (11.98b)} \xrightarrow{\text{Gauss}} \quad 0 = \delta_\varepsilon S \doteq \int \mathrm{d}^4x \, \frac{\delta \mathcal{L}}{\delta g^{\mu\nu}} \delta_\varepsilon g^{\mu\nu} \tag{11.99}$$

with ⁂ *variational derivative*

$$\boxed{\frac{\delta \mathcal{L}}{\delta g^{\mu\nu}} := \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - \partial_\lambda \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}_{,\lambda}} \,.} \tag{11.100}$$

vi | We now want to find an explicit expression for the variation $\delta_\varepsilon g^{\mu\nu}$ of the metric:

a | The metric transforms as a $(2, 0)$ tensor:

$$\bar{g}^{\mu\nu}_\varepsilon(\bar{x}_\varepsilon) \overset{11.85b}{=} \frac{\partial \bar{x}^\mu_\varepsilon}{\partial x^\alpha} \frac{\partial \bar{x}^\nu_\varepsilon}{\partial x^\beta} g^{\alpha\beta}(x) \tag{11.101a}$$

$$\overset{11.90}{=} g^{\mu\nu}(x) + \varepsilon^\mu{}_{,\alpha} g^{\alpha\nu}(x) + \varepsilon^\nu{}_{,\beta} g^{\mu\beta}(x) + \mathcal{O}\left(\varepsilon^2\right) \tag{11.101b}$$

We dropped higher powers in the variation ε^μ and its derivatives. Note that we implicitly assume that derivatives of ε^μ are also infinitesimal; this is a restriction on reasonably smooth variations ε^μ (which we are free to impose).

b | On the other hand, we can also simply expand the new metric:

$$\bar{g}^{\mu\nu}_\varepsilon(\bar{x}_\varepsilon) = \bar{g}^{\mu\nu}_\varepsilon(x + \varepsilon) = \bar{g}^{\mu\nu}_\varepsilon(x) + g^{\mu\nu}{}_{,\lambda} \varepsilon^\lambda + \mathcal{O}\left(\varepsilon^2\right) \tag{11.102}$$

In the second term we replaced $\bar{g}$ by g because their difference is of order ε which, together with the ε^λ, can be absorbed in $\mathcal{O}\left(\varepsilon^2\right)$.

c | Eqs. (11.101) and (11.102) →

$$\delta_\varepsilon g^{\mu\nu}(x) = \bar{g}^{\mu\nu}_\varepsilon(x) - g^{\mu\nu}(x) = \varepsilon^\mu{}_{,\lambda} g^{\lambda\nu} + \varepsilon^\nu{}_{,\lambda} g^{\lambda\mu} - \varepsilon^\lambda g^{\mu\nu}{}_{,\lambda} \tag{11.103}$$

d | We can use covariant derivatives and metric-compatibility to simplify this expression:

Eqs. (10.50), (10.56) and (10.74) →

$$\delta_\varepsilon g^{\mu\nu} \overset{\circ}{=} \varepsilon^\mu{}_{;\lambda} g^{\lambda\nu} + \varepsilon^\nu{}_{;\lambda} g^{\lambda\mu} \tag{11.104}$$

This is the variation of the metric under the infinitesimal diffeomorphism φ_ε.

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5 | Eq. (11.99) $\xrightarrow{\text{Eq. (11.104)}}$ (Use the symmetry of $g^{\mu\nu}$.)

$$0 \doteq \int d^4x \frac{\delta\mathcal{L}}{\delta g^{\mu\nu}} \left[\varepsilon^{\mu}{}_{;\lambda} g^{\lambda\nu} + \varepsilon^{\nu}{}_{;\lambda} g^{\lambda\mu}\right] = \int \underbrace{d^4x\sqrt{g}}_{\text{Scalar}} \underbrace{\frac{2}{\sqrt{g}}\frac{\delta\mathcal{L}}{\delta g^{\mu\nu}}}_{\to\text{Tensor}} \underbrace{\varepsilon^{\mu}{}_{;\lambda} g^{\lambda\nu}}_{\text{Tensor}} \tag{11.105}$$

This motivates the definition of the … (recall the scalar Lagrangian $L \equiv \frac{\mathcal{L}}{\sqrt{g}}$)

⁂ *(Hilbert) Energy-Momentum Tensor (HEMT)*:

$$T_{\mu\nu} := \frac{2}{\sqrt{g}}\frac{\delta\mathcal{L}}{\delta g^{\mu\nu}} = \frac{2}{\sqrt{g}}\frac{\delta(\sqrt{g}L)}{\delta g^{\mu\nu}} \tag{11.106}$$

This tensor is always symmetric: $T_{\mu\nu} = T_{\nu\mu}$.

¡! At this point it is unclear what $T_{\mu\nu}$ has to do with energy and momentum, and whether it relates to the EMT/BRT derived in Sections 6.3.1 and 6.3.2 from Noether's first theorem.

6 | With this new notation, we have:

$$0 \doteq \int d^4x\sqrt{g}\, T_\mu{}^\lambda \varepsilon^\mu{}_{;\lambda} = \int d^4x\sqrt{g}\Big[\underbrace{\left(\varepsilon^\mu T_\mu{}^\lambda\right)_{;\lambda}}_{\text{Surface term}} - \varepsilon^\mu T_\mu{}^\lambda{}_{;\lambda}\Big] \tag{11.107}$$

Here we used the Leibniz product rule for covariant derivatives in reverse.

Compact variation $\varepsilon^\mu(x) \xrightarrow{\text{Eq. (10.103)}}$

$$\int d^4x\sqrt{g}\,\varepsilon^\mu T_\mu{}^\lambda{}_{;\lambda} \doteq 0 \tag{11.108}$$

Valid for all *local* variations $\varepsilon^\mu(x) \to T_\mu{}^\lambda{}_{;\lambda} \doteq 0$

→ (use that we can pull indices up under the covariant derivative)

$$T^{\mu\lambda}{}_{;\lambda} \doteq 0 \tag{11.109}$$

Thus the covariant divergence of the HEMT vanishes for solutions ϕ of the matter EOMs and an arbitrary metric g.

- Eq. (11.109) is the generally covariant form of the current conservation Eq. (6.92) that follows from translation invariance for the (symmetric = Belinfante) energy-momentum tensor: $T^{\mu\lambda}{}_{,\lambda} = 0$. This suggests that the name "energy-momentum tensor" is warranted, though the exact relation between the previously defined BRT Eq. (6.106) is as of yet unclear. See also the → *next* two points.
- ¡! If you think of it, Eq. (11.109) looks like free lunch! *We didn't specify any special properties of the matter theory* $S_g[\phi]$ – except for its general covariance, which, as argued in Section 9.2, is physically vacuous. This immediately implies that it cannot be possible to derive conserved quantities from Eq. (11.109) in general because these would be conserved in *any* reasonable theory of (fundamental) physics, independent of its symmetries!

  We will study this in more detail in → Section 11.5.

- Diffeomorphism invariance is a local symmetry in that its transformations can affect fields on compact regions of spacetime only; this makes it a *gauge symmetry* (→ *Hole argument*). Noether's *first* theorem applies to *global* symmetries and guarantees the existence of a conserved charge for each generator of the symmetry. By contrast, Noether's *second* theorem applies to *local* (gauge) symmetries (like diffeomorphism invariance) and implies the existence of *constraint equations* that reduce the number of degrees of freedom constrained by the Euler-Lagrange equations. From this angle, Eq. (11.109) is a product of Noether's *second* theorem; in particular, it does not come with conserved Noether charges (which come with Noether's *first* theorem).

7 | Useful relations: (➔ Problemset 4)

The following relations are useful to compute the HEMT $T_{\mu\nu}$ for a specific Lagrangian L:

- With Eq. (10.91) it follows

$$\frac{\partial(\sqrt{g})}{\partial g_{\mu\nu}} = \frac{1}{2}\sqrt{g}g^{\mu\nu}\,. \tag{11.110}$$

  Note that this is a derivative wrt. $g_{\mu\nu}$ and not $g^{\mu\nu}$!

- Because of $g^{\mu\rho}g_{\rho\nu} = \delta^\mu_\nu$ one finds for the derivative

$$\frac{\partial g_{\alpha\beta}}{\partial g^{\mu\nu}} \overset{\circ}{=} -g_{\alpha\mu}g_{\beta\nu} \tag{11.111}$$

  and with this

$$\frac{\partial(\sqrt{g})}{\partial g^{\mu\nu}} = \frac{\partial(\sqrt{g})}{\partial g_{\alpha\beta}}\frac{\partial g_{\alpha\beta}}{\partial g^{\mu\nu}} = -\frac{1}{2}\sqrt{g}g_{\mu\nu} \tag{11.112}$$

  Note the additional minus!

8 | Example: Maxwell theory:

Details: ➔ Problemset 4

i | Recall the scalar Lagrangian Eq. (6.56) of electrodynamics on curved spacetimes:

$$L(\partial A, g) = -\frac{1}{16\pi}F_{\alpha\beta}F_{\lambda\rho}g^{\lambda\alpha}g^{\rho\beta} \quad \text{with} \quad F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha\,. \tag{11.113}$$

ii | The variational derivative Eq. (11.100) simplifies because L is independent of derivatives of the metric:

$$\frac{\delta(\sqrt{g}L)}{\delta g^{\mu\nu}} \overset{11.100}{=} \frac{\partial(\sqrt{g}L)}{\partial g^{\mu\nu}} \overset{11.112}{=} -\frac{1}{2}\sqrt{g}g_{\mu\nu}L + \frac{1}{8\pi}\sqrt{g}F_{\mu\lambda}F^\lambda{}_\nu \tag{11.114}$$

iii | Eq. (11.106) $\overset{\circ}{\rightarrow}$

$$\boxed{T_{\mu\nu} = \frac{1}{4\pi}\left[F_{\mu\lambda}F^\lambda{}_\nu + \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right]} \tag{11.115}$$

- This is indeed the "old" Belinfante-Rosenfeld energy momentum tensor (BRT) of the electromagnetic field discussed in Eq. (6.110), only for an arbitrary metric $g_{\mu\nu}$ in arbitrary coordinates instead of the Minkowski metric $\eta_{\mu\nu}$ in inertial coordinates. This suggests that the Hilbert energy momentum tensor is the generally covariant generalization of the symmetric BRT to arbitrary spacetimes.

- One can show rigorously that the Hilbert energy momentum tensor (HEMT), defined above, and the symmetric Belinfante-Rosenfeld energy momentum tensor (BRT), defined in Section 6.3.2, lead to the same expressions [78]. So our result for electrodynamics is no coincidence.
- Therefore one can use Eq. (11.106) as an alternative to compute the symmetric energy momentum tensor for relativistic theories on Minkowski space (as an alternative to the symmetrization procedure discussed in Section 6.3.2). To do so, use the MCP to make a Lorentz invariant Lagrangian generally covariant, then assume that the coordinates are arbitrary for the computation, and specialize to inertial coordinates at the end (in which the metric takes the form $\eta_{\mu\nu}$).

9 | Example: Klein-Gordon field:

Details: ➔ Problemset 4

i | ⊲ Real Klein-Gordon field ϕ, Eq. (11.37):

$$L(\phi,\partial\phi,g) = \frac{1}{2}g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - \frac{m^2}{2}\phi^2 \tag{11.116}$$

Note that this is a scalar Lagrangian; the covariant derivatives equal partial derivatives because ϕ is a scalar: $\partial_\mu\phi = \phi_{,\mu} = \phi_{;\mu} = \nabla_\mu\phi$.

ii | The variational derivative Eq. (11.100) simplifies because L is again independent of derivatives of the metric:

$$\frac{\delta(\sqrt{g}L)}{\delta g^{\mu\nu}} \overset{11.100}{=} \frac{\partial(\sqrt{g}L)}{\partial g^{\mu\nu}} \overset{11.112}{=} -\frac{1}{2}\sqrt{g}g_{\mu\nu}L + \frac{1}{2}\sqrt{g}(\partial_\mu\phi)(\partial_\nu\phi) \tag{11.117}$$

iii | Eq. (11.106) $\overset{\circ}{\to}$

$$T_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\left(\phi^{,\alpha}\phi_{,\alpha} - m^2\phi^2\right) \tag{11.118}$$

This is the HEMT of the real Klein-Gordon field.

iv | Let us verify that it satisfies Eq. (11.109):

$$T^{\mu\nu}{}_{;\nu} = (\phi^{,\mu}\phi^{,\nu})_{;\nu} - \frac{1}{2}g^{\mu\nu}\left(\phi^{,\alpha}\phi_{,\alpha} - m^2\phi^2\right)_{;\nu} \tag{11.119a}$$

$$= \phi^{,\mu}\phi^{,\nu}{}_{;\nu} \underbrace{+\phi^{,\mu}{}_{;\nu}\phi^{,\nu} - g^{\mu\nu}\phi^{,\alpha}\phi_{,\alpha;\nu}}_{\overset{10.53}{=}0} + m^2\phi^{,\mu}\phi \tag{11.119b}$$

$$= \phi^{,\mu}\underbrace{\left(\phi^{,\nu}{}_{;\nu} + m^2\phi\right)}_{\overset{11.38}{=}0} \doteq 0\,. \tag{11.119c}$$

Here we used $g^{\mu\nu}{}_{;\rho} = 0$ since the connection is metric-compatible. Note that we had to invoke the equation of motion in the last step to show that the divergence vanishes.

11.5. ‡ Killing vector fields and conservation laws

As noted previously, Eq. (11.109) cannot be used to define conserved quantities in general. Here we discuss this in more detail and characterize the conditions under which conserved quantities actually do exist.

Killing (vector) fields are named after German mathematician ↑ *Wilhelm Karl Joseph Killing* (1847–1923). Note that this is a situation were case sensitivity is of paramount importance: *Killing fields* and *the* ↑ *Killing Fields* are not related whatsoever; luckily, we're only concerned with the former.

1 | Problem: (Compare this to Eq. (6.85) which implies Eq. (6.87).)

$$T^{\mu\nu}{}_{;\nu} \overset{10.57a}{=} T^{\mu\nu}{}_{,\nu} + \Gamma^{\nu}{}_{\rho\nu} T^{\mu\rho} + \Gamma^{\mu}{}_{\rho\nu} T^{\rho\nu} \overset{11.109}{\doteq} 0 \tag{11.120a}$$

$$\overset{10.92}{\Longleftrightarrow} \quad \frac{1}{\sqrt{g}} \left(\sqrt{g} T^{\mu\nu}\right)_{,\nu} + \underbrace{\Gamma^{\mu}{}_{\rho\nu} T^{\rho\nu}}_{\text{No surface term} ☹} \doteq 0 \tag{11.120b}$$

→ Cannot be integrated by Gauss to define conserved charge!

→ *Question:* Are there conditions under which conserved quantities can be defined?

2 | Let $\varepsilon^\mu(x) \equiv \varepsilon\, \xi^\mu(x)$ with $\varepsilon \ll 1$; ⊲ Eq. (11.104) and demand

$$\delta_\varepsilon g^{\mu\nu} = \varepsilon \left(\xi^\mu{}_{;\lambda} g^{\lambda\nu} + \xi^\nu{}_{;\lambda} g^{\lambda\mu} \right) \overset{!}{=} 0 \tag{11.121}$$

For a given metric, this is a differential equation to be solved for vector fields $\xi^\mu(x)$.

Pull indices down →

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0 \quad ⁂\ \textit{Killing equation} \qquad \Leftrightarrow\ \xi^\mu(x) \quad ⁂\ \textit{Killing (vector) field} \tag{11.122}$$

- Whether the Killing Eq. (11.122) has solutions depends on the metric $g_{\mu\nu}$; it can have none / one / multiple solutions. One then says that the metric has no / one / multiple Killing fields.
- At this point it is unclear how Killing fields can help us to find conserved quantities. However, it is intuitively clear what Killing fields are: They generate a continuous group of diffeomorphisms that do not change the metric! Such special diffeomorphisms are called ↑ *isometries* of the Riemannian manifold; i.e., the infinitesimal generators of the isometry group of a Riemannian manifold are the Killing fields. This group obviously depends on the geometry of the manifold, and thus its metric. Killing fields therefore characterize the *symmetries* of a spacetime manifold. A generic spacetime will have "bumps" and "twists", and therefore no symmetries/Killing fields.
- Example: Minkowski space:

  On Minkowski space we can use global inertial coordinates, so that the covariant derivatives become partial derivatives everywhere:

  $$\xi_{\mu,\nu} + \xi_{\nu,\mu} = 0\,. \tag{11.123}$$

  It is straightforward to check that the most general solution reads

  $$\xi_\mu(x) = a_\mu + b_{\mu\nu} x^\nu \quad \text{with} \quad b_{\mu\nu} = -b_{\nu\mu} \tag{11.124}$$

  with a_μ and $b_{\mu\nu}$ arbitrary integration constants. There are 4 linearly independent solutions parametrized by a^μ, and 6 solutions parametrized by the antisymmetric $b_{\mu\nu}$; so in total Minkowski space has 10 Killing vector fields. These correspond to 4 *translations* (the a_μ-solutions), 3 spatial rotations and 3 boosts (the $b_{\mu\nu}$-solutions). The isometric diffeomorphisms of Minkowski space are the ← *Poincaré transformations*! Recall our discussion of the Lorentz group in Section 4.3.
- One can show that a 3 + 1-dimensional spacetime can have at most 10 Killing fields. In general, a D-dimensional Riemannian manifold can have at most $\frac{1}{2}D(D+1)$ Killing fields; such manifolds are called ↑ *maximally symmetric spaces*. Minkowski space is an example of a maximally symmetric space. For more details: ↑ CARROLL [102] (§3.8 & §3.9, pp. 133–144).

3 | Conserved quantities:

We show now that Killing fields can be used to construct conserved quantities:

i | $\lessdot$ Spacetime with Killing vector field ξ^μ → Define the 4-current

$$J_\xi^\mu := T^{\mu\nu}\xi_\nu\,. \tag{11.125}$$

Then it follows for the covariant divergence of this current:

$$\frac{1}{\sqrt{g}}\left(\sqrt{g}J_\xi^\mu\right)_{,\mu} \overset{10.95}{=} J^\mu_{\xi;\mu} = (T^{\mu\nu}\xi_\nu)_{;\mu} \overset{11.122}{=} \xi_\nu T^{\mu\nu}{}_{;\mu} \underset{11.109}{\doteq} 0 \tag{11.126}$$

Here we used that the contraction of a symmetric with an antisymmetric tensor vanishes.

ii | Then the charge

$$Q_\xi := \int \mathrm{d}^3x\,\sqrt{g}J_\xi^0 \quad \text{is conserved:} \quad \frac{\mathrm{d}Q_\xi}{\mathrm{d}x^0} \doteq 0\,. \tag{11.127}$$

This is true if $J_\xi^i = 0$ on the boundary of space; i.e., if one considers closed systems.

Proof. From Eq. (11.126) we have

$$0 \doteq \int \mathrm{d}^3x\left(\sqrt{g}J_\xi^\mu\right)_{,\mu} = \int \mathrm{d}^3x\left(\sqrt{g}J_\xi^0\right)_{,0} + \int \mathrm{d}^3x\left(\sqrt{g}J_\xi^i\right)_{,i} \tag{11.128}$$

and therefore

$$\frac{\mathrm{d}Q_\xi}{\mathrm{d}x^0} = \int \mathrm{d}^3x\left(\sqrt{g}J_\xi^0\right)_{,0} \doteq -\int \mathrm{d}^3x\left(\sqrt{g}J_\xi^i\right)_{,i} \overset{\text{Gauss}}{=} -\int_\partial \mathrm{d}\sigma_i\,\sqrt{g}J_\xi^i = 0\,. \tag{11.129}$$

Here we used the assumption that the current vanishes on the spatial surface ∂: $J_\xi^i = 0$. ■

iii | For the special case of point mechanics, one finds a more explicit expression:

$\lessdot$ Free particle described by the equation of motion:

$$\frac{\mathrm{D}u^\mu}{\mathrm{D}\tau} = 0 \quad \text{with 4-velocity } u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}. \tag{11.130}$$

→ With the Killing vector ξ^μ it follows:

$$\frac{\mathrm{d}(\xi_\mu u^\mu)}{\mathrm{d}\tau} = \frac{\mathrm{D}(\xi_\mu u^\mu)}{\mathrm{D}\tau} = \underbrace{u^\mu\frac{\mathrm{D}\xi_\mu}{\mathrm{D}\tau}}_{=0} + \underbrace{\xi_\mu\frac{\mathrm{D}u^\mu}{\mathrm{D}\tau}}_{=0} \overset{\circ}{=} 0 \tag{11.131}$$

The second summand vanishes because of Eq. (11.130) and the first one because of the Killing Eq. (11.122):

$$u^\mu\frac{\mathrm{D}\xi_\mu}{\mathrm{D}\tau} \overset{10.49}{=} u^\mu\xi_{\mu;\nu}u^\nu = 0\,. \tag{11.132}$$

Here we used that the Killing equation tells us that $\xi_{\mu;\nu}$ is an antisymmetric tensor and that the contraction of a symmetric with an antisymmetric tensor vanishes.

→ Conserved quantity:

$$\xi_\mu u^\mu = \text{const} \tag{11.133}$$

This can be useful to integrate the geodesic equation on symmetric spacetimes.

4 | Stationary & Static spacetimes:

Using Killing fields, we can define two special classes of spacetimes with useful properties:

i | We can define the following special class of metrics g:

$$g \text{ is ⁂ } \textit{stationary} \quad :\Leftrightarrow \quad g \text{ has (asymptotically) } \textit{time-like} \text{ Killing vector}$$

"Asymptotically time-like" means that there is a Killing vector field that becomes time-like at infinity. It is possible that the Killing field becomes space-like in some finite region of space.

ii | $\lessdot$ Time-like Killing field ξ^μ (i.e., $\xi^\mu \xi_\mu > 0$)

→ Choose *w.l.o.g.* coordinates where $\xi^\mu = (1, 0, 0, 0)$

This means that we choose the x^0-axis such that it points along the Killing field.

→

$$\text{Killing Eq. (11.122)} \quad \stackrel{\text{Eq. (11.103)}}{\Longleftrightarrow} \quad \frac{\partial g^{\mu\nu}}{\partial x^0} = 0 \tag{11.134}$$

→ $g_{\mu\nu}(x) = g_{\mu\nu}(\vec{x})$ independent of the time-like coordinate $x^0 \equiv t$!

→ In these coordinates, a stationary metric has the general form:

$$\mathrm{d}s^2 = g_{00}(\vec{x})\mathrm{d}t^2 + g_{0i}(\vec{x})\mathrm{d}t\,\mathrm{d}x^i + g_{i0}(\vec{x})\mathrm{d}x^i\,\mathrm{d}t + g_{ij}(\vec{x})\mathrm{d}x^i\,\mathrm{d}x^j \tag{11.135}$$

- *Interpretation:* A *stationary* spacetime "does exactly the same thing at every time" [102].
- *Example:* The ↑ *Kerr metric* of a rotating black hole is stationary.

iii | There is an interesting subclass of stationary spacetimes:

$\lessdot$ Stationary metric g. *Iff* there exists a coordinate system such that $g_{0i}(\vec{x}) = 0$,

$$\mathrm{d}s^2 = g_{00}(\vec{x})\mathrm{d}t^2 + g_{ij}(\vec{x})\mathrm{d}x^i\,\mathrm{d}x^j \tag{11.136}$$

the metric is called ⁂ *static*.

- *Interpretation:* A *static* spacetime "doesn't do anything at all" [102].
- *Example:* The → *Schwarzschild metric* of a non-rotating black hole is static.
- The feature that distinguishes a stationary (non-static) metric Eq. (11.135) from a static metric Eq. (11.136) is that the latter is invariant under reversal $t \mapsto -t$ of the time-like coordinate.

  This makes sense if you compare the static Schwarzschild metric (non-rotating black hole) with the stationary Kerr metric (rotating black hole): If you invert time, a non-rotating black hole looks exactly the same, so also its metric should not change; hence it has the form Eq. (11.136) and is static. By contrast, the angular momentum of a rotating Kerr black hole changes sign under $t \mapsto -t$, so that it looks different in a time-reversed world; hence its metric should change as well, which necessitates the off-diagonal terms of a stationary (non-static) metric Eq. (11.135). (Note that the Kerr black hole still describes a *stationary* system in that its angular momentum doesn't change in time, and consequently the metric "does the same thing at every time".)

iv | In summary:

$$\textit{Static}\text{ spacetime} \quad \overset{\Rightarrow}{\not\Leftarrow} \quad \textit{Stationary}\text{ spacetime}$$

- *Counterexample:* The ↑ *Kerr metric* is stationary but not static.