↓Lecture 23 [04.06.24]

11.2. Classical mechanics in the gravitational field

We now apply the MCP to obtain a formulation of the classical mechanics of points on a given spacetime with Lorentzian metric $g_{\mu\nu}$.

1 | $\sphericalangle$ *Free* particle of mass m

→ EOM in local inertial system according to EEP and SPECIAL RELATIVITY (← Eq. (5.46)):

$$m\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = 0 \tag{11.44}$$

$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$: 4-velocity (in local inertial coordinates)
τ: Proper time of particle

Note that you can multiply the mass m on the left-hand side, but it cancels anyway! This reflects the fact that, in SPECIAL RELATIVITY, the trajectory of a free particle is independent of its (inertial) mass.

2 | In local inertial coordinates it is $\Gamma^\mu{}_{\nu\rho} = 0 \rightarrow$

$$\text{Eq. (11.44)} \overset{\Gamma=0}{\Leftrightarrow} \underbrace{m\frac{\mathrm{D}u^\mu}{\mathrm{D}\tau} \overset{10.37}{=} m\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} + m\Gamma^\mu{}_{\nu\rho}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau} = 0}_{\text{Generally covariant geodesic equation}} \tag{11.45}$$

→ Eq. (11.45) valid in *arbitrary coordinates* (GRP) on *arbitrary spacetimes* (MCP / EEP)!

With this we mean that the MCP suggests that the correct equation of motion for a free particle on an arbitrary (potentially curved) spacetime of GENERAL RELATIVITY is Eq. (11.45), i.e., the ← *geodesic equation*!

→

> In GENERAL RELATIVITY *free particles* follow *geodesics* in spacetime.

It is important to fully grasp what just happened (the procedure is deceptively simple but subtle):

i | We know from SPECIAL RELATIVITY that Eq. (11.44) describes the movement of free particles correctly (and globally) on flat Minkowski space. The Christoffel symbols of the Minkowski metric in (globally) inertial coordinates vanish *everywhere*, so that Eq. (11.44) is trivially equivalent to Eq. (11.45). But Eq. (11.45) is the (unique) generally covariant tensor equation that reduces to Eq. (11.44) in (globally) inertial coordinates. Eq. (11.45) still describes the physics of a free, relativistic particle on Minkowski space, but now in *arbitrary* coordinates. To reiterate: As long as you fix the metric of spacetime as the Minkowski metric (which therefore plays the role of a ← *background*), Eq. (11.45) is simply a more general (but equivalent) formulation of classical mechanics in SPECIAL RELATIVITY, i.e., there is no new physics contained in the equation!

What you witnessed is the transition from a *coordinate-specific* formulation of a physical model to a *generally covariant* formulation. This is the principle of general relativity GRP in action, and, as we discussed in Section 9.2, it is physically vacuous (in particular, it is not specific to GENERAL RELATIVITY). Nonetheless we already gained something: Since Eq. (11.45) holds in arbitrary coordinates, you can use this equation to obtain the relativistic equations of motion in curvilinear coordinates (e.g., accelerated Rindler coordinates, ➔ Problemset 3). Just compute the Christoffel symbols of the Minkowski metric in these coordinates, and you are good to go!

ii | GENERAL RELATIVITY enter the stage in the next step (which is purely interpretational since Eq. (11.45) is already given and does not need to be modified):

The `EEP` claims that, even in the presence of gravity, the laws of SPECIAL RELATIVITY remain valid *locally*. Thus Eq. (11.44) must be valid in every *local* inertial frame of a potentially curved spacetime. Eq. (11.45) implements this demand because, for every point of spacetime and in *locally* inertial coordinates, the equation reduces to Eq. (11.44). That this happens is the motivation behind the `MCP`.

[Note that the local inertial coordinates x^μ are different from point to point! That is, astronauts in different space stations all can use Eq. (11.44) to describe free moving particles in their small labs, but their coordinate systems are not the same (and typically not even overlapping).]

The physically non-trivial claim, coming from `EEP` and built into `MCP`, is now that Eq. (11.45) describes the trajectories of free particles correctly on *all spacetimes* (and not only on flat Minkowski space). This claim is far from vacuous as it makes empirical statements about the motion of free particles in the presence of gravity (= curvature), something that SPECIAL RELATIVITY had nothing to say about. Whether Eq. (11.45) is correct in the presence of gravity is an empirical claim (as the validity of the `EEP` is) that needs to be tested experimentally.

Notes:

- We can rewrite Eq. (11.45) in the form

$$m \underbrace{\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau}}_{\text{“4-acceleration”}} = \underbrace{-m\Gamma^\mu{}_{\nu\rho}u^\nu u^\rho}_{\text{“4-force”}}, \tag{11.46}$$

  which suggests the interpretation of the right-hand side as the “gravitational 4-force” acting on the test particle. The connection coefficients $\Gamma^\mu{}_{\nu\rho}$ then play the role of the “gravitational field strength” and (since $\Gamma \propto \partial g$) the metric $g_{\mu\nu}(x)$ can be identified as the “gravitational potential”. In the Newtonian limit (→ *below*) this identification is indeed reproduced.

  However, use these identifications with a grain of salt; the whole point of GENERAL RELATIVITY is to identify the effect of gravity as spacetime curvature (which we will finally do in Chapter 12), and not as a classical force (which can be present in addition to gravity, → *below*). Note also that the “4-acceleration” in Eq. (11.46) is a *coordinate* acceleration and *not* a tensor, i.e., it cannot be identified with a physical acceleration [this is in contrast to the 4-acceleration in SPECIAL RELATIVITY, ← Eq. (4.49)].

  The reason is that the coordinates x^μ in the definition of u^μ are *arbitrary*; in particular, they do *not* convey metric information on their own (recall our discussion of the role played by coordinates in Section 9.2). Hence the “4-force” on the right-hand side (which is also not a tensor!) does not correspond to a coordinate-independent, physical force; it is a fictitious “coordinate force”, similar to the fictitious Coriolis force in classical mechanics (which is purely a consequence of a particular choice of coordinates). There is a difference, though: In Newtonian mechanics you can always find a coordinate system (corresponding to an inertial frame) where the Coriolis force vanishes everywhere. By contrast, the “4-force” in Eq. (11.46) can only be made vanish *locally* (in locally geodesic coordinates, that is) but not globally (if this is possible, the Christoffel symbols vanish everywhere and spacetime is flat).

- Please appreciate how elegant the implementation of the universality of free fall (= equivalence of gravitational and inertial mass, `WEP`) in this formalism is: In Eq. (11.45) there is only *one* place to put a mass (in front of the absolute derivative). Only when we write the absolute derivative as sum of two terms, this single mass starts to play two (seemingly) different roles, namely that of *inertial* mass on the left-hand side of Eq. (11.46), and that of *gravitational* mass on the right-hand side. But the two are necessarily identical, a fact that Newtonian mechanics cannot explain! It is this natural emergence of the WEP that corroborates (and historically motivated) a *metric* theory of gravity.

- Non-minimal coupling:

  It is instructive to study what happens if we ignore the MCP and produce a non-minimally coupled, generally covariant equation. For example, we could postulate the following equation that (supposedly) describes the motion of a free particle in GENERAL RELATIVITY:

$$m\frac{\mathrm{D}u^\mu}{\mathrm{D}\tau} \overset{?}{=} \xi R_{;\sigma}u^\sigma u^\mu \tag{11.47}$$

  with $\xi \in \mathbb{R}$ some coupling constant. This equation …

  - … is generally covariant (→ implements GRP).
  - … reduces to Eq. (11.44) on flat Minkowski space (→ consistent with SPECIAL RELATIVITY).

  The problem is that, on a generic curved spacetime, the curvature-related *tensor* $R_{;\sigma}$ does *not* vanish in locally inertial coordinates, so that Eq. (11.47) takes the locally inertial form

$$m\frac{\mathrm{d}\bar{u}^\mu}{\mathrm{d}\tau} = \xi \bar{R}_{;\sigma}\bar{u}^\sigma \bar{u}^\mu\,, \tag{11.48}$$

  which does *not* reproduce the physics of SPECIAL RELATIVITY [namely Eq. (11.44)], and therefore violates the EEP. Had we adhered to the MCP, we would have never added the curvature term in the first place, and this violation would not occur.

3 | External forces:

In SPECIAL RELATIVITY we not only discussed free particles but also ones that are acted upon by some external force [← Eq. (5.6)]. Using the MCP we immediately obtain the generally covariant form of our relativistic equation of motion:

$$m\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = K^\mu \quad \xrightarrow{\text{MCP}} \quad m\frac{\mathrm{D}u^\mu}{\mathrm{D}\tau} = m\left(\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu{}_{\nu\rho}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau}\right) = K^\mu \tag{11.49}$$

Here K^μ is a placeholder for some force that transforms as a contravariant tensor and acts locally on the particle. We will find an explicit example → *later* when we discuss the electromagnetic field.

→

Forces make trajectories *deviate* from geodesics. (11.50)

For example, the force pushing you into your seat right now is the phenomenological consequence of *not* following a geodesic in spacetime. (A geodesic trajectory corresponds to free fall, but your seat is in the way!)

4 | Some relations:

i | $u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$ is a tensor

u^μ is not a tensor *field* as it is only defined along the trajectory $x^\mu(\tau)$, but $u^\mu\partial_\mu \in TM$ so that it transforms as a $(1,0)$ tensor.

→ $\|u\|^2 = g_{\mu\nu}u^\mu u^\nu$ is a scalar:

$$\|\dot{x}\|^2 = g_{\mu\nu}u^\mu u^\nu = \{g_{\mu\nu}u^\mu u^\nu\}^{\text{LI}} = \eta_{\mu\nu}\bar{u}^\mu\bar{u}^\nu \underset{\text{EEP}}{\overset{4.48}{=}} c^2 > 0 \tag{11.51}$$

(LI = Locally inertial coordinates)

→ Physical trajectories $x^\mu(\tau)$ of massive particles must be *time-like*!

This is the generalization of Eq. (4.48).

ii | With this we find:

$$0 = \frac{\mathrm{d}c^2}{\mathrm{d}\tau} = \frac{\mathrm{D}(g_{\mu\nu}u^\mu u^\nu)}{\mathrm{D}\tau} = 2g_{\mu\nu}u^\nu\frac{\mathrm{D}u^\mu}{\mathrm{D}\tau} \tag{11.52}$$

Here we used the metric-compatibility of the Levi-Civita connection Eq. (10.74) and the product rule for covariant/absolute derivatives. →

$$\frac{\mathrm{D}u^\mu}{\mathrm{D}\tau}u_\mu = 0 \tag{11.53}$$

This is the generally covariant analog of Eq. (4.50).

iii | Eq. (11.49) $\xrightarrow{\text{Eq. (11.53)}}$

$$u_\mu K^\mu = 0 \tag{11.54}$$

This means that the 4-velocity u^μ of a physical trajectory is always orthogonal to the 4-acceleration $\frac{\mathrm{D}u^\mu}{\mathrm{D}\tau}$ and hence the 4-force K^μ.

iv | The ← *4-momentum* is defined as previously:

$$p^\mu = mu^\mu \tag{11.55}$$

Eq. (11.51) →

$$\|p\|^2 = g_{\mu\nu}p^\mu p^\nu = m^2c^2 \tag{11.56}$$

This is the generalization of Eq. (5.4).

5 | Variational principle (for a free particle):

As usual, the equation of motion can be found via a variational principle from an action. Since Eq. (11.45) is generally covariant, the Lagrangian must be a scalar. An application of the MCP to the action Eqs. (5.41) and (5.43) of a free particle in SPECIAL RELATIVITY immediately yields the correct result:

$$\begin{matrix}\text{Eq. (5.41)}\\ \text{Eq. (5.43)}\end{matrix} \xrightarrow{\text{MCP}} S_g[x] = -mc\int_x \mathrm{d}s = -mc\int_x \sqrt{g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu} \tag{11.57}$$

with

$$\delta S_g[x] \overset{!}{=} 0 \quad\Leftrightarrow\quad \text{Eq. (11.45)} \tag{11.58}$$

We do not need to prove this! This is exactly the variation that we used to *derive* the geodesic equation (which we now interpret as the equation of motion for a free particle!); ← Section 10.3.3.

6 | Newtonian approximation:

i | ⊲ Non-relativistic particle in a *Newtonian* gravitational potential $\phi = -\frac{MG}{r}$:

$$L = -mc^2 + \frac{1}{2}mv^2 - m\phi \tag{11.59}$$

→ Non-relativistic (≈) action:

$$S_g \approx \int \mathrm{d}t\, L = -mc \int \mathrm{d}t \left(c - \frac{v^2}{2c} + \frac{\phi}{c}\right) \tag{11.60}$$

To understand where Lagrangian & action come from, recall Eq. (5.42):

$$S_\eta = -mc^2 \int \mathrm{d}t\,\sqrt{1 - v^2/c^2} \approx -mc^2 \int \mathrm{d}t \left(1 - \frac{v^2}{2c^2}\right) \tag{11.61}$$

→ Non-relativistic approximation of Lagrangian in SPECIAL RELATIVITY:

$$L = -mc^2 + \frac{1}{2}mv^2 \tag{11.62}$$

Above we simply added an additional Newtonian gravitational potential.

ii | Identify the line element in the fully relativistic action:

$$\text{Eqs. (11.57) and (11.60)} \quad \rightarrow \quad \mathrm{d}s \approx \left(c - \frac{v^2}{2c} + \frac{\phi}{c}\right)\mathrm{d}t \tag{11.63}$$

Use $\mathrm{d}\vec{x} = \vec{v}\mathrm{d}t$ and drop terms $\propto v^2/c^2$ (slow particle) and $\propto \phi^2/c^4$ (weak field) $\overset{\circ}{\rightarrow}$

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = (\mathrm{d}s)^2 \approx \left(1 + \frac{2\phi}{c^2}\right)(c\,\mathrm{d}t)^2 - (\mathrm{d}\vec{x})^2 \tag{11.64}$$

This allows us to identify the Newtonian potential as 00-component of the metric tensor:

$$g_{00} \approx 1 + \frac{2\phi}{c^2} \quad \text{with} \quad \phi = -\frac{MG}{r} \tag{11.65}$$

- Note that this result is consistent with our previous interpretation of the metric as the analog of a gravitational potential in GENERAL RELATIVITY.
- This result demonstrates that the dominant effect of a weak gravitational field is the modification of the *time*-component of the metric, i.e., a modification of the tick-rate of clocks as a function of height (→ *gravitational time dilation*).

11.3. Electrodynamics in the gravitational field

Now we use the MCP to generalize ← *classical electrodynamics* to curved spacetimes:

1 | Remember: (Section 6.2)

- Field strength tensor: $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$

- *Homogeneous* Maxwell equations: (Eqs. (6.42) and (6.50a))

$$\tilde{F}^{\mu\nu}{}_{,\nu} = 0 \quad \Leftrightarrow \quad F_{\langle\mu\nu,\lambda\rangle} = F_{\mu\nu,\lambda} + F_{\nu\lambda,\mu} + F_{\lambda\mu,\nu} = 0 \tag{11.66}$$

  These equations are *identities* if $F_{\mu\nu}$ is expressed in terms of a gauge field A_μ.

- *Inhomogeneous* Maxwell equations: (Eq. (6.50b))

$$F^{\mu\nu}{}_{,\nu} = -\frac{4\pi}{c} j^\mu \tag{11.67}$$

2 | The field strength Lorentz tensor can be generalized to a proper tensor via the MCP:

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \quad \xrightarrow{\text{MCP}} \quad F_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu} \stackrel{\circ}{=} A_{\nu,\mu} - A_{\mu,\nu} \tag{11.68}$$

This follows from the symmetry of the Christoffel symbols.

→ No covariant derivatives needed!

Put differently: Our old field strength *Lorentz* tensor was a proper $(0,2)$ tensor all along!

3 | Homogeneous Maxwell equations (HME):

The homogeneous Maxwell equations follow directly with the MCP:

$$\text{Eq. (11.66)} \quad \xrightarrow{\text{MCP}} \quad F_{\langle\mu\nu;\lambda\rangle} \stackrel{\circ}{=} F_{\langle\mu\nu,\lambda\rangle} = 0 \tag{11.69}$$

→ The HME have the same form as in SPECIAL RELATIVITY

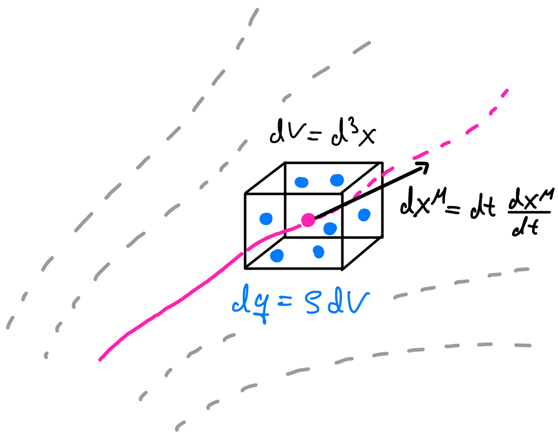
- If $F_{\mu\nu}$ is expressed in terms of the gauge field A_μ, this is again an *identity*, i.e., it is true for all gauge fields A_μ and hence does not impose constraints on A_μ. To see this without calculations, note that $F_{\mu\nu}$ does not contain connection coefficients due to Eq. (11.68). This means that in locally geodesic coordinates we have immediately $\{F_{\langle\mu\nu;\lambda\rangle}\}^{\text{LG}} = F_{\langle\mu\nu,\lambda\rangle} = 0$; since $F_{\langle\mu\nu;\lambda\rangle}$ is a tensor, it is $F_{\langle\mu\nu;\lambda\rangle} = 0$ in all coordinate systems. As this line of reasoning never imposes any constraint on A_μ, Eq. (11.69) is an identity.
- In coordinate-free notation, the homogeneous Maxwell equations read $\mathrm{d}F = 0$, with the 2-form $F = \mathrm{d}A$ and the 1-form A (gauge connection); ← Eq. (6.70). That Eq. (11.69) is an identity simply follows from $\mathrm{d}\mathrm{d}A = 0$ since $\mathrm{d}^2 = 0$ for the exterior derivative. The fact that all connection coefficients drop out, and the equations do not depend on the metric, is reflected by the fact that $\mathrm{d}F = 0$ is a well-defined expression on any differentiable manifold – neither connection nor metric required (e.g., in form of a Hodge dual).

  Note that the equations are completely identical to ← Eq. (6.70), where we discussed the coordinate-free notation in the context of SPECIAL RELATIVITY. This emphasizes once more that general covariance is not a characteristic feature of GENERAL RELATIVITY.

4 | Current:

Before we can discuss the inhomogeneous Maxwell equation, we must revisit the charge current:

i | *Remember* (Section 6.2): Charge $\mathrm{d}q = \rho\,\mathrm{d}^3x$ in volume $\mathrm{d}V = \mathrm{d}^3x$ is scalar quantity:



$$\rho\,\mathrm{d}^3x = \bar\rho\,\mathrm{d}^3\bar x \quad\Rightarrow\quad \underbrace{\underbrace{\rho\,\mathrm{d}^3x}_{\text{Scalar}}\;\underbrace{\mathrm{d}x^\mu}_{4-\text{vector}}}_{4-\text{vector}} = \rho\,\mathrm{d}^3x\,\mathrm{d}t\,\frac{\mathrm{d}x^\mu}{\mathrm{d}t} = \underbrace{\frac{\sqrt{g}}{c}\,\mathrm{d}^4x}_{\substack{\text{Eq. (10.101)}\\ \Downarrow\\ \text{Scalar}}}\;\underbrace{\frac{\rho}{\sqrt{g}}\frac{\mathrm{d}x^\mu}{\mathrm{d}t}}_{\substack{\Downarrow\\ 4-\text{vector}}} \tag{11.70}$$

Recall that not d^4x but $\sqrt{g}\mathrm{d}^4x$ transforms as a scalar [Eq. (10.101)]; in SPECIAL RELATIVITY, we only considered Lorentz transformations (which have $g = 1$) so that we didn't have to make this distinction.

This implies that the 4-current must be defined as follows to be a contravariant vector:

$$J^\mu := \frac{\rho}{\sqrt{g}}\frac{\mathrm{d}x^\mu}{\mathrm{d}t} \overset{6.18}{=} \frac{j^\mu}{\sqrt{g}} \tag{11.71}$$

ii | Charge conservation is encoded by the covariant continuity equation:

$$J^\mu{}_{;\mu} \overset{\circ}{=} 0 \qquad \text{(Continuity equation)} \tag{11.72}$$

To show Eq. (11.72), use Eqs. (10.95) and (11.71) to rewrite the covariant divergence as

$$J^\mu{}_{;\mu} \overset{10.95}{=} \frac{1}{\sqrt{g}}\left(\sqrt{g}J^\mu\right)_{,\mu} \overset{11.71}{=} \frac{1}{\sqrt{g}}j^\mu{}_{,\mu}\,. \tag{11.73}$$

At every point we can transform into locally inertial coordinates where we know that

$$\{J^\mu{}_{;\mu}\}^{\mathrm{LI}} = j^\mu{}_{,\mu} \underset{\text{EEP}}{\overset{6.24}{=}} 0 \tag{11.74}$$

Because $J^\mu{}_{;\mu}$ is a scalar, Eq. (11.72) follows in all coordinate systems.

5 | Inhomogeneous Maxwell equations (IME):

We can now use the MCP to construct the IME valid on arbitrary spacetimes:

$$\begin{matrix}\text{Eq. (11.67)}\\ \text{Eq. (11.71)}\end{matrix} \xrightarrow{\text{MCP}} F^{\mu\nu}{}_{;\nu} \overset{10.96}{=} \frac{1}{\sqrt{g}}\left(\sqrt{g}F^{\mu\nu}\right)_{,\nu} = -\frac{4\pi}{c}J^\mu \tag{11.75}$$

- Using the form of the covariant divergence in the middle (which follows from Eq. (10.96) for an antisymmetric tensor), it is easy to verify that in locally inertial coordinates the special relativistic form Eq. (11.67) is recovered so that the EEP is satisfied. (To show this, use that in locally inertial coordinates first derivatives of the metric vanish.)
- In contrast to the HME in Eq. (11.69), the IME Eq. (11.75) are *not* identical to their Lorentz covariant counterparts Eq. (11.67) but true covariant extensions thereof. In particular, the metric makes an appearance in the equations. This means that, because of the IME, classical electrodynamics is not a *topological* but a *geometrical* field theory, in that its solutions depend on the geometry of spacetime. This is not surprising: One would expect the solutions for the electromagnetic field to be different if space were a sphere, for example. Put differently, the electromagnetic field reacts in a non-trivial way to curvature in spacetime. As we want a theory that reproduces the observed deflection of light in the vicinity of heavy masses (← Section 8.2), and we would like gravity to be completely encoded in the metric, this is certainly nice to see!
- That the current must satisfy the continuity equation Eq. (11.72) for Eq. (11.75) to have solutions is straightforward to show in a local inertial frame:

$$-\frac{4\pi}{c}\left\{J^{\mu}{}_{;\mu}\right\}^{\mathrm{LI}} \overset{11.75}{=} \left[\frac{1}{\sqrt{g}}\left(\sqrt{g}F^{\mu\nu}\right)_{,\nu}\right]_{,\mu} = \frac{1}{\sqrt{g}}\left(\sqrt{g}F^{\mu\nu}\right)_{,\nu,\mu} = 0\,. \tag{11.76}$$

  Here we used that in locally inertial coordinates first derivatives of the metric vanish and *partial* derivatives commute, together with the antisymmetry of $F_{\mu\nu}$. (Note that you cannot – without additional input – conclude that $F^{\mu\nu}{}_{;\nu;\mu} = 0$ since covariant derivatives in general do not commute! Here this is true because $F_{\mu\nu}$ is antisymmetric, as shown above.)
- In coordinate-free notation, also the IME looks the same as in SPECIAL RELATIVITY:

$$\star\mathrm{d}(\star F) \overset{*}{=} J \quad \Leftrightarrow \quad \text{Eq. (6.70b)} \tag{11.77}$$

  with the 1-form $J = \frac{4\pi}{c} J_\mu \mathrm{d}x^\mu$ and the 2-form $F = \frac{1}{2} F_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$. To derive this, one must use the definition of the Hodge star operator on arbitrary pseudo-Riemannian manifolds to show that $\star\mathrm{d}(\star F) \overset{*}{=} F_\mu{}^\nu{}_{;\nu}\,\mathrm{d}x^\mu$.

  The fact that the IME knows about the metric is reflected by the Hodge star operator in Eq. (11.77) (which is defined via the metric). That the equation looks the same as in SPECIAL RELATIVITY might be surprising at first, but this is the whole point of the MCP: the coupling to gravity is postulated to be *minimal* – and what is more minimal than not changing the equation at all? (Beware: That the equation looks the same does not mean that the EM field does not couple to the metric! What changed between SPECIAL RELATIVITY and GENERAL RELATIVITY is that, previously, the metric to define the Hodge star was *fixed* as the Minkowski metric, now it is a *dynamical* field with its own dynamics.)

6 | Action:

The covariant action of electrodynamics follows via the MCP from the old Maxwell action, and by replacing the old current by the new contravariant one:

$$\text{Eq. (6.56)} \xrightarrow[\text{Eq. (11.71)}]{\text{MCP}} S_g[A] = \int \mathrm{d}^4x\, \mathcal{L}(A, \partial A, g) = \int \mathrm{d}^4x \sqrt{g}\left[-\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} - \frac{1}{c}J^\mu A_\mu\right] \tag{11.78}$$

Note that the metric $g_{\mu\nu}$ is also hidden in the two contractions between the field strength tensors!

$\overset{\circ}{\rightarrow}$ Euler-Lagrange Equations:

$$\frac{\partial \mathcal{L}}{\partial A_\mu} - \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} = 0 \quad \Leftrightarrow \quad F^{\mu\nu}{}_{;\nu} = -\frac{4\pi}{c} J^\mu \tag{11.79}$$

7 | Using the action it is possible to trace the continuity equation (= charge conservation) back to the invariance of the action under gauge transformations of the form $\tilde{A}_\mu = A_\mu + \partial_\mu \lambda$. To this end, consider the local gauge variation $\delta A_\mu = \partial_\mu \lambda$, generated by a compactly supported scalar $\lambda(x)$ (meaning: $\lambda(x)$ vanishes everywhere except for a finite region of spacetime), and compute the variation of the action:

$$\delta S_g = -\frac{1}{c}\int \mathrm{d}^4x \sqrt{g} J^\mu \partial_\mu \lambda = -\underbrace{\frac{1}{c}\int \mathrm{d}^4x \partial_\mu \left(\sqrt{g} J^\mu \lambda\right)}_{\overset{\text{Gauss}}{=} 0} + \frac{1}{c}\int \mathrm{d}^4x \lambda \partial_\mu \left(\sqrt{g} J^\mu\right) . \tag{11.80}$$

Here we used that $\delta F_{\mu\nu} = 0$ since $F_{\mu\nu}$ is gauge invariant (this is true whether or not A_μ extremizes the action). The first summand on the right vanishes because $\lambda(x)$ is compactly supported and vanishes on the surface the integration volume.

If A_μ solves the IME, and therefore extremizes the action, the variation vanishes: $\delta S_g = 0$. Since this must be true for arbitrary compactly supported $\lambda(x)$, the continuity equation follows:

$$\partial_\mu \left(\sqrt{g} J^\mu\right) = 0 \quad \Leftrightarrow \quad J^\mu{}_{;\mu} = 0 . \tag{11.81}$$

8 | Charged particle in an electromagnetic field:

It is now straightforward to write down a generally covariant equation that describes the motion o a charged particle in an electromagnetic field in an arbitrary gravitational field (= metric $g_{\mu\nu}$).

Recall Section 6.4 →

$$\text{Eq. (6.130)} \xrightarrow[\text{Eq. (11.49)}\\ \text{Eq. (11.68)}]{\text{MCP}} \frac{\mathrm{D}p^\mu}{\mathrm{D}\tau} = \frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} + m\Gamma^\mu{}_{\nu\rho} u^\nu u^\rho = \frac{q}{c} F^\mu{}_\nu u^\nu \tag{11.82}$$

Here we used the definition of the particle momentum Eq. (11.55).

From our discussions in Sections 11.2 and 11.3, it is clear that this covariant equation reduces to Eq. (6.130) in locally inertial coordinates, and thus obeys the EEP.

9 | Energy-momentum tensor:

The symmetric ← *Belinfante-Rosenfeld energy-momentum tensor* (BRT) of the general covariant theory Eq. (11.78) follows immediately:

$$\text{Eq. (6.110)} \xrightarrow{\text{MCP}} T^{\mu\nu}_{\text{em}} = \frac{1}{4\pi}\left[g^{\mu\alpha} F_{\alpha\beta} F^{\beta\nu} + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}\right] \tag{11.83}$$

This tensor will describe the effect of energy and momentum carried by the electromagnetic field on the gravitational field (metric) of GENERAL RELATIVITY; i.e., Eq. (11.83) shows up on the right-hand side of the → *Einstein field equations* as a source of gravity.