↓**Lecture 21** [14.05.24]

6 | Corollaries:

- Working with a metric-compatible connection has the benefit that one can pull indices up and down *within* a covariant derivative:

$$T_{i;k} = (g_{im}T^m)_{;k} = \underbrace{g_{im;k}}_{=0} T^m + g_{im}T^m{}_{;k} \overset{10.74}{=} g_{im}T^m{}_{;k} \tag{10.83}$$

- The inverse metric is also covariantly constant:

$$g^{ik}{}_{;l} = 0 \tag{10.84}$$

To show this, note that $\delta^i_{j;l} = 0$ [Eq. (10.57b)] and use the Leibniz product rule:

$$0 = \delta^i_{j;l} = (g^{ik}g_{kj})_{;l} = g^{ik}{}_{;l}\,g_{kj} + g^{ik}g_{kj;l} \overset{10.74}{=} g^{ik}{}_{;l}\,g_{kj}\,. \tag{10.85}$$

7 | Local inertial coordinates: (Details: ➔ Problemset 2)

i | ⊲ Levi-Civita connection in ← *locally geodesic coordinates* at $p \in M$:

(For simplicity, we assume that the point p has the coordinates $u(p) = 0$.)

$$\partial_l g_{ik}(0) \overset{10.75}{=} \underbrace{\Gamma^m{}_{il}(0)}_{=0}\, g_{mk} + \underbrace{\Gamma^m{}_{kl}(0)}_{=0}\, g_{im} = 0 \tag{10.86}$$

→ In these coordinates, the metric tensor is constant in linear order:

$$g_{ij}(x) = g_{ij}(0) + \frac{1}{2}\partial_\alpha\partial_\beta g_{ij}(0)\,x^\alpha x^\beta + \mathcal{O}(x^3) \tag{10.87}$$

ii | ⊲ *Affine* coordinate transformation: $\bar{x}^i = M^i{}_j\,x^j + b^i \xrightarrow{\text{Eq. (10.39)}} \bar{\Gamma}^i{}_{kl} = 0$

Note that under affine/linear coordinate transformations, the connection coefficients transform like tensors! In particular, if the connection coefficients vanish in one (geodesic) coordinate system, they vanish in all coordinates that can be reached by affine transformations; i.e., geodesic coordinates are not unique!

→ Use linear transformation to bring metric of signature (r, s) into the form

$$\bar{g}_{ij}(0) = \mathrm{diag}(\underbrace{+1,\dots,+1}_{\times r},\underbrace{-1,\dots,-1}_{\times s})\,. \tag{10.88}$$

That this is possible follows from ↑ *Sylvester's law of inertia*: First, use the symmetry of the metric to diagonalize the matrix $g_{ij}(0)$ by an orthogonal transformation, then use another non-singular transformation to normalize the eigenvalues to ± 1.

iii | ⊲ Special case $(r = 1, s = 3)$ = Lorentzian manifold →

Metric in ⁂ *locally inertial coordinates*:

$$\boxed{\bar{g}_{\mu\nu}(\bar{x}) \overset{\bar{x}\to 0}{\approx} \eta_{\mu\nu} + \frac{1}{2}\partial_\alpha\partial_\beta\bar{g}_{\mu\nu}(0)\,\bar{x}^\alpha\bar{x}^\beta} \tag{10.89}$$

- In words: For every point of a Lorentzian manifold there exist coordinate systems such that the metric in this point takes the Minkowski form $\eta_{\mu\nu}$ and is constant in linear order; we call such charts *locally inertial coordinates*.
- Recall that Lorentz transformations are linear and leave the Minkowski metric invariant [← *Eq.* (4.21)]. This implies that locally inertial coordinates are also not unique: You can use arbitrary Lorentz transformations without changing the structure of Eq. (10.89).

8 | Useful relations:

Here we list a few identities that will be useful for many calculations in GENERAL RELATIVITY. You prove these relations in ➔ Problemset 2.

- The trace of the Christoffel symbols simplifies to

$$\Gamma^i{}_{ki} \overset{\circ}{=} \frac{1}{2} g^{im} g_{im,k} \,. \tag{10.90}$$

- With the determinant of the metric $g = \det(g_{im})$, the *inverse* metric can be written as

$$g^{im} \overset{\circ}{=} \frac{1}{g} \frac{\partial g}{\partial g_{im}} \,. \tag{10.91}$$

- With Eqs. (10.90) and (10.91), the trace of the Christoffel symbols takes the simple form

$$\Gamma^i{}_{ki} = \frac{1}{2g} g_{,k} = \left(\ln \sqrt{\pm g}\right)_{,k} \,, \tag{10.92}$$

  such that $\pm g > 0$.

  Note: In GENERAL RELATIVITY it is $\det(g_{\mu\nu}) < 0$ (because of the Lorentzian signature) and we redefine $g := -\det(g_{\mu\nu}) > 0$ to simplify expressions.
- The other trace of the Christoffel symbols can also be written in a compact form:

$$g^{kl} \Gamma^i{}_{kl} \overset{\circ}{=} -\frac{1}{\sqrt{g}} \left(\sqrt{g} g^{im}\right)_{,m} \,. \tag{10.93}$$

- It is straightforward to show the following useful identity:

$$g_{ik} (g^{kl})_{,m} \overset{\circ}{=} -(g_{ik})_{,m} \, g^{kl} \,. \tag{10.94}$$

- The ⁂ *covariant divergence* of a contravariant vector field is defined as one would expect:

$$A^i{}_{;i} \overset{10.92}{=} A^i{}_{,i} + A^l (\ln \sqrt{g})_{,l} = \frac{1}{\sqrt{g}} \left(\sqrt{g} A^i\right)_{,i} \tag{10.95}$$

- For the covariant divergence of an *antisymmetric* $(2,0)$-tensor there is a similar expression:

$$A^{ik}{}_{;k} \overset{\circ}{=} \frac{1}{\sqrt{g}} \left(\sqrt{g} A^{ik}\right)_{,k} \quad \text{with} \quad A^{ij} = -A^{ji} \,. \tag{10.96}$$

- Eq. (10.95) can be used to rewrite the covariant Laplacian (divergence of a gradient) of a scalar:

$$\Delta\phi \equiv \phi^{;i}{}_{;i} = \frac{1}{\sqrt{g}} \left(\sqrt{g} g^{ik} \phi_{,k}\right)_{,i} \,. \tag{10.97}$$

  The differential operator Δ maps scalar functions onto scalar functions and is known as ↑ *Laplace-Beltrami operator*.

- Generalized divergence theorem:

  i | $\lhd$ Coordinate transformation $\bar{x} = \varphi(x)$

  $\rightarrow$ D-dimensional (oriented) volume element (more precisely: volume form) transforms as (← *Eq.* (3.39))

$$\mathrm{d}^D\bar{x} = \det\left(\frac{\partial\bar{x}}{\partial x}\right)\mathrm{d}^D x \tag{10.98}$$

  with ↓ *Jacobian determinant* $\det\left(\frac{\partial\bar{x}}{\partial x}\right)$.

  ii | The determinant of the metric transforms in the opposite way (← *Eq.* (3.54)):

$$\sqrt{\bar{g}} = \left|\det\left(\frac{\partial x}{\partial\bar{x}}\right)\right|\sqrt{g} \tag{10.99}$$

  (Note the absolute value of the Jacobian determinant!)

  iii | Hence the product of metric determinant and (oriented) volume element transforms like a pseudo scalar:

$$\sqrt{\bar{g}}\,\mathrm{d}^D\bar{x} = \mathrm{sign}\left[\det\left(\frac{\partial\bar{x}}{\partial x}\right)\right]\sqrt{g}\,\mathrm{d}^D x\,. \tag{10.100}$$

  Here $\mathrm{sign}\left[\det\left(\frac{\partial\bar{x}}{\partial x}\right)\right]$ denotes the sign of the Jacobian determinant, which encodes whether the coordinate transformation is orientation preserving $(+1)$ or not (-1). This makes $\sqrt{g}\,\mathrm{d}^D x$ transform like a *pseudo* scalar.

  If we are only interested in non-oriented volume elements, or restrict ourselves to orientation-preserving coordinate transformations, Eq. (10.100) simplifies to a true scalar transformation:

$$\sqrt{\bar{g}}\,\mathrm{d}^D\bar{x} = \sqrt{g}\,\mathrm{d}^D x\,. \tag{10.101}$$

  This subtlety will not be important in the following and we use Eq. (10.101) henceforth.

  iv | Eq. (10.101) is the reason why integrals over scalar quantities $\bar{\phi}(\bar{x}) = \phi(x)$ are form-invariant under arbitrary coordinate transformations if we use the "modified" volume element $\sqrt{g}\,\mathrm{d}^D x$ for integration:

$$\boxed{\int \underbrace{\underbrace{\mathrm{d}^N x\sqrt{g(x)}}_{\text{Scalar}}\;\underbrace{\phi(x)}_{\text{Scalar}}}_{\text{Scalar}} \stackrel{\bar{x}=\varphi(x)}{=} \int \mathrm{d}^N\bar{x}\sqrt{\bar{g}(\bar{x})}\,\bar{\phi}(\bar{x})} \tag{10.102}$$

  v | Using the covariant divergence Eq. (10.95) and the modified volume element Eq. (10.101), we find the generalized form of the divergence theorem

$$\boxed{\int_V \mathrm{d}^D x\sqrt{g}\,A^i{}_{;i} \stackrel{10.95}{=} \int_V \mathrm{d}^D x\,\partial_i\left(\sqrt{g}A^i\right) \stackrel{\text{Gauss}}{=} \oint_{\partial V}\mathrm{d}\sigma_i\sqrt{g}\,A^i\,,} \tag{10.103}$$

  where ∂V is the surface of V and $\mathrm{d}\sigma_i$ denotes the $D-1$-dimensional surface element.

10.3.2. The RIEMANN curvature tensor

Now that we identified the special Levi-Civita connection (which can be computed from the metric), we can also express its curvature tensor (then called *Riemann* curvature tensor) in terms of the metric as well:

Detailed calculations: ➔ Problemset 3

9 | ⊲ Locally geodesic coordinates LG:

$$\{R_{iklm}\}^{\mathrm{LG}} = \{g_{ia} R^a{}_{klm}\}^{\mathrm{LG}} \overset{10.70}{=} g_{ia}\left(\partial_l \Gamma^a{}_{km} - \partial_m \Gamma^a{}_{kl}\right) \tag{10.104}$$

Recall that the connection coefficients – but not their derivatives – vanish in these coordinates!

10 | Now use the explicit form of the Levi-Civita connection to find an expression in terms of the metric:

$$\boxed{\{R_{iklm}\}^{\mathrm{LG}} \overset{10.79}{=} \frac{1}{2}\left(g_{im,k,l} + g_{kl,i,m} - g_{il,k,m} - g_{km,i,l}\right)} \tag{10.105}$$

- Recall that $g_{ij,k} = 0$ in locally geodesic coordinates [← *Eq.* (10.86)].
- This expressions tells us that curvature prevents us from finding coordinates in which the *second* derivatives of the metric vanish.

11 | In general coordinates, the expression becomes more complicated:

$$\boxed{R_{iklm} \overset{\circ}{=} \{R_{iklm}\}^{\mathrm{LG}} + g_{ab}\left(\Gamma^a{}_{kl}\Gamma^b{}_{im} - \Gamma^a{}_{km}\Gamma^b{}_{il}\right)} \tag{10.106}$$

To show this, start from Eqs. (10.70) and (10.79) and use Eqs. (10.75) and (10.94).

12 | Algebraic identities:

- Eqs. (10.105) and (10.106) →

$$R_{iklm} = -R_{kilm}\,, \quad R_{iklm} = -R_{ikml}\,, \quad R_{iklm} = R_{lmik} \tag{10.107}$$

In words: the Riemann tensor is *antisymmetric* in the first two and last two indices, but *symmetric* if both pairs of indices are swapped.

- ⁂ *First/Algebraic* BIANCHI *identity*:

The *cyclic sums* of Riemann tensors vanish identically:

$$\boxed{R_{i\langle klm\rangle} \equiv R_{iklm} + R_{ilmk} + R_{imkl} \overset{\circ}{=} 0} \tag{10.108}$$

The same is true for the cyclic sums of arbitrary triples of indices.

The relations Eqs. (10.107) and (10.108) are *identities*, i.e., their validity follows directly from the definition of the Riemann curvature tensor, independent of the specific metric. This means that a Riemann tensor in D-dimensions has less independent components as the naïve count D^4 suggests.

For example, on the $D = 4$-dimensional spacetime of GENERAL RELATIVITY, at most 20 (and not $4^4 = 256$) numbers are needed to specify R_{iklm} in every point of the spacetime manifold (➔ Problemset 3). [Beware: This does not mean that there are 20 *physical* degrees of freedom in GENERAL RELATIVITY! R_{iklm} is still a tensor and can be modified by arbitrary coordinate transformations without changing its physical content. We will see → *later* that GENERAL RELATIVITY has a large gauge group (→ *diffeomorphism invariance*) so that there are way less physical degrees of freedom than the 20 alluded to above.]

13 | ⁂ *Second/Differential* Bianchi *identity*:

The cyclic sums of covariant derivatives of the Riemann tensor vanish identically:

$$\boxed{R^a{}_{k\langle lm;n\rangle} \equiv R^a{}_{klm;n} + R^a{}_{kmn;l} + R^a{}_{knl;m} \mathrel{\hat{=}} 0} \tag{10.109}$$

Proof. A neat trick to prove tensor relations is to choose a coordinate system in which their derivation is simple, and then use the tensor character of the involved objects to infer the validity of the relation in general coordinates.

Both the Riemann tensor and covariant derivatives are particularly simple in locally geodesic coordinates:

$$\left\{R^a{}_{klm;n}\right\}^{\text{LG}} \overset{10.70}{\underset{10.56}{=}} \Gamma^a{}_{km,l,n} - \Gamma^a{}_{kl,m,n} \,. \tag{10.110}$$

Adding up the cyclic permutations of this expression yields:

$$\left\{R^a{}_{k\langle lm;n\rangle}\right\}^{\text{LG}} = \left\{R^a{}_{klm;n}\right\}^{\text{LG}} + \left\{R^a{}_{kmn;l}\right\}^{\text{LG}} + \left\{R^a{}_{knl;m}\right\}^{\text{LG}} \tag{10.111a}$$

$$= \Gamma^a{}_{km,l,n} - \Gamma^a{}_{kl,m,n} + \Gamma^a{}_{kn,m,l} - \Gamma^a{}_{km,n,l} + \Gamma^a{}_{kl,n,m} - \Gamma^a{}_{kn,l,m} \tag{10.111b}$$

$$= 0 \tag{10.111c}$$

Now, since $R^a{}_{k\langle lm;n\rangle}$ is a tensor and vanishes in one coordinate system, it vanishes in *all* coordinate systems (because tensor components transform linearly under coordinate transformations); thus $R^a{}_{k\langle lm;n\rangle} = 0$ and we are done. ■

Notes:

- Remember that commutators $[A,B] = AB - BA$ satisfy the ↓ *Jacobi identity*:

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0 \,. \tag{10.112}$$

But the ← *Ricci identity* Eq. (10.71) relates the curvature tensor (not necessarily a Riemannian one, but the connection must be torsion-free) to the commutator of covariant derivatives:

$$A_{k[;l;m]} = A_a R^a{}_{klm} \,. \tag{10.113}$$

Using this, one can derive the second (and also the first) Bianchi identity from the Jacobi identity; see Nakahara [133] (p. 269).

14 | Derived tensors:

The following tensors can be derived from the Riemann tensor and will play an important role in the formulation of GENERAL RELATIVITY:

i | The only non-trivial contraction of the Riemann tensor sums one index of the first pair with one index of the second pair (all other contractions vanish due to symmetries):

$$\boxed{\text{⁂ Ricci } \textit{tensor:} \quad R_{kl} := R^a{}_{kla} = -R^a{}_{kal}} \tag{10.114}$$

ii | The Ricci tensor is symmetric:

$$R_{kl} = R_{lk} \tag{10.115}$$

To show this, contract the first Bianchi identity Eq. (10.108),

$$R^a{}_{kla} + R^a{}_{lak} + R^a{}_{akl} = 0\,, \tag{10.116}$$

and use $R^a{}_{akl} = 0$ due to the antisymmetry of the Riemann tensor.

→ In $D = 4$ dimensions, the Ricci tensor has 10 algebraically independent components.

iii | We can contract the Ricci tensor to obtain a curvature scalar:

$$\boxed{\text{⁂ Ricci } \textit{scalar:} \quad R := g^{ab} R_{ab} = R^a{}_a} \tag{10.117}$$

iv | ⁂ *Contracted* BIANCHI *identity*:

Ricci tensor and -scalar obey an identity that derives from the second Bianchi identity:

$$\boxed{R^a{}_{n;a} = \frac{1}{2} R_{;n}} \tag{10.118}$$

Proof. To show this, contract the differential Bianchi identity Eq. (10.109) over a and m:

$$R_{kl;n} - R_{kn;l} - R_k{}^a{}_{nl;a} = 0\,. \tag{10.119}$$

Tracing out k and l (recall that our connection is metric-compatible, i.e., we are allowed to pull indices up/down inside covariant derivatives) yields:

$$0 = g^{kl} R_{kl;n} - g^{kl} R_{kn;l} - g^{kl} R_k{}^a{}_{nl;a} \tag{10.120a}$$

$$\overset{10.117}{=} R_{;n} - R^l{}_{n;l} - R^{la}{}_{nl;a} \tag{10.120b}$$

$$\overset{10.114}{=} R_{;n} - R^l{}_{n;l} - R^a{}_{n;a} \tag{10.120c}$$

$$= R_{;n} - 2R^a{}_{n;a}\,. \tag{10.120d}$$

■

v | As preparation for GENERAL RELATIVITY, we define another tensor using the Ricci tensor, Ricci scalar, and metric:

$$\boxed{\text{⁂ Einstein } \textit{tensor:} \quad G_{ij} := R_{ij} - \frac{1}{2} g_{ij} R} \tag{10.121}$$

For $D = 4$ on a Lorentzian manifold, this tensor will be used as the left-hand side of the → *Einstein field equations*.

vi | The form of Eq. (10.121) is structurally similar to the contracted Bianchi identity. Indeed, Eq. (10.118) immediately implies:

$$\boxed{\text{Eq. (10.118)} \quad \Rightarrow \quad G^a{}_{i;a} = 0} \tag{10.122}$$

- Eq. (10.122) will be crucial for the consistency of the → *Einstein field equations* with energy momentum conservation.

- For $D = 4$ one can show that the Einstein tensor $G_{\mu\nu}$ (besides the metric tensor $g_{\mu\nu}$) is the *only* rank-2 tensor with vanishing (covariant) divergence that one can construct from the metric and its first and second derivatives [134, 135]. This result is known as ↑ *Lovelock's theorem* and states under which conditions the field equations of GENERAL RELATIVITY (including the cosmological constant) are *unique* (→ *later*). The uniqueness of $G_{\mu\nu}$ and Lovelock's theorem impose important constraints on possible extensions (or modifications) of GENERAL RELATIVITY.

10.3.3. Geodesics

In Section 10.2 we defined "straight lines" as curves that keep their direction constant, and formalized this notion as ← *autoparallel curves*. Now that we have a metric at hand, we can also define "straight lines" as the *shortest* curves connecting two points. We will show now that these two concepts coincide for the metric-compatible, torsion-free Levi-Civita connection induced by the metric:

15 | ⊲ Length of curve γ connecting two points P_2 and P_2 [← *Eq.* (3.55)]:

$$L[\gamma] = \int_\gamma \mathrm{d}s = \int_{\lambda_1}^{\lambda_2} \mathrm{d}\lambda \sqrt{g_{ij}\dot{x}^i\dot{x}^j} \tag{10.123}$$

Here, $x^i(\lambda_{1/2})$ are the coordinates of $P_{1/2}$ in some chart. The right expression is independent of both the parametrization $x^i(\lambda)$ of the curve and the coordinate system.

To see the latter, recall that for a coordinate transformation $\bar{x} = \varphi(x)$ it is

$$\frac{\mathrm{d}\bar{x}^i}{\mathrm{d}\lambda} = \frac{\partial\bar{x}^i}{\partial x^m}\frac{\mathrm{d}x^m}{\mathrm{d}\lambda} \quad \text{and} \quad \bar{g}_{ij} = \frac{\partial x^k}{\partial\bar{x}^i}\frac{\partial x^l}{\partial\bar{x}^j}g_{kl}\,. \tag{10.124}$$

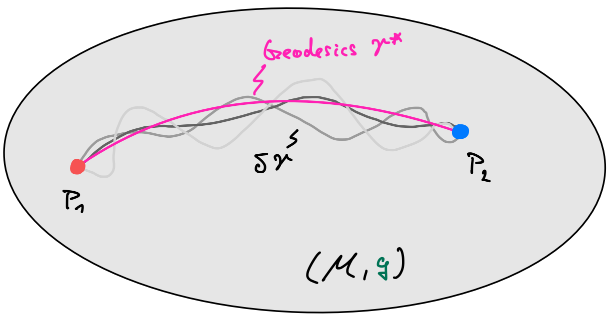
Remember that the directional derivatives $\dot{x}^i\partial_i$ along a curve are vectors in the tangent space T_pM and transform accordingly. Thus, in the expression Eq. (10.123), the total derivative wrt. λ is important! In contrast to the special coordinate transformations of SPECIAL RELATIVITY (Lorentz transformations), the coordinates x^i themselves do *not* transform as tensors (they transform like $\bar{x} = \varphi(x)$, which is non-linear in general).

16 | "Straight line" from P_1 to P_2 ≡ Shortest curve γ^* (⁂ *Geodesics*) from P_1 to P_2

¡! Strictly speaking, we will not study *globally shortest* curves, but curves that *locally extremize* the length functional Eq. (10.123). For now, you can think of geodesics as "shortest curve" connecting two points, but keep in mind that this is not necessarily true (→ *comments below*).

→ Extremize length over curves starting at P_1 and terminating at P_2:

$$\delta L = \delta\int_{P_1}^{P_2} \mathrm{d}s \stackrel{!}{=} 0 \tag{10.125}$$



17 | $\sphericalangle$ Strictly monotonic, differentiable function χ & Class of "Lagrangians"

$$\mathfrak{L}_\chi(x,\dot{x}) := \chi\big(\underbrace{g_{kl}(x)\dot{x}^k\dot{x}^l}_{=:y}\big) \tag{10.126}$$

For example: $\chi(x) = \sqrt{x}$ yields the integrand of Eq. (10.123) as Lagrangian.

→ More general variation principle:

$$\delta \int_{P_1}^{P_2} \mathrm{d}\lambda\, \mathfrak{L}_\chi(x,\dot{x}) = 0 \tag{10.127}$$

Depending on χ, this "action" is no longer reparametrization invariant in general.

18 | → Euler-Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial \mathfrak{L}_\chi}{\partial \dot{x}^i}\right) - \frac{\partial \mathfrak{L}_\chi}{\partial x^i} = 0 \quad \Leftrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\chi'(y)2g_{ik}\dot{x}^k\right) - \chi'(y)\frac{\partial g_{kl}}{\partial x^i}\dot{x}^k\dot{x}^l = 0 \tag{10.128}$$

19 | $\sphericalangle$ Parametrization with $y = g_{ij}(x)\dot{x}^i\dot{x}^j \equiv \|\dot{x}\|_x^2 \overset{!}{=} 1 = \text{const}$

This choice fixes an *affine parametrization* $\lambda = s$ of the curve γ where the "velocity" $\|\dot{x}\|_x$ is constant. Since we require $\|\dot{x}\|_x = 1$, the "time" λ is equal to the length s of the curve from the start to $x^i(\lambda)$ (up to a constant offset).

Later, on the (pseudo-Riemannian) Lorentzian manifolds of GENERAL RELATIVITY, we will also consider *space-like* geodesics with $y < 0$; for such curves, you must add an additional minus in the square root of Eq. (10.123) and choose $y = -1 = \text{const}$ instead. The rest of the derivation is then completely analogous.

→ $\chi'(y) = \text{const} \neq 0$ (strict monotonic!) →

$$\text{Eq. (10.128)} \quad \Leftrightarrow \quad g_{ik}\ddot{x}^k + g_{ik,l}\dot{x}^k\dot{x}^l - \frac{1}{2}g_{kl,i}\,\dot{x}^k\dot{x}^l = 0 \tag{10.129}$$

Note that this differential equation is independent of χ!

$$\text{Eq. (10.129)} \quad \Leftrightarrow \quad g_{ik}\ddot{x}^k + \underbrace{\frac{1}{2}\left(g_{il,k} + g_{ik,l} - g_{kl,i}\right)}_{\Gamma_{ikl}}\dot{x}^k\dot{x}^l = 0\,. \tag{10.130}$$

20 | Identify Christoffel symbols Eq. (10.79) $\overset{\circ}{\rightarrow}$

$$\boxed{\frac{\mathrm{d}^2x^i}{\mathrm{d}\lambda^2} + \Gamma^i{}_{kl}\frac{\mathrm{d}x^k}{\mathrm{d}\lambda}\frac{\mathrm{d}x^l}{\mathrm{d}\lambda} = 0 \qquad ⁂\ \textit{Geodesic equation}} \tag{10.131}$$

Solutions of this DGL are called ⁂ *(affinely parametrized) Geodesics*.

21 | Notes:

- ¡! We derived the Geodesic equation by a variational principle *extremizing* the length between two points. This means that geodesics are not necessarily the *shortest* curves between two points. Ignoring the peculiarities of *pseudo*-Riemannian metrics for now (→ Section 11.1), Geodesics are only *locally* the shortest connections between close by points, but not necessarily *globally*. Put differently: Every shortest path connecting two points is a geodesic, but not every geodesic connecting two points is a shortest path.

An example is a great circle on a sphere connecting two points (→ *below*), say the north pole and a point on the equator. The great circle satisfies the geodesic equation everywhere, and is therefore a geodesic. The shortest path connecting the two points is part of the great circle (and therefore also a geodesic). But the "long way around" is certainly not the shortest path (but still a geodesic, as it is also part of the great circle).

- With our derivation we showed that the curves (Geodesics) that solve the geodesic equation Eq. (10.131) not only extremize the *length* Eq. (10.123), but the more general class of "actions" defined by the "Lagrangian" Eq. (10.126). This will be useful → *later* when we study the classical mechanics of points on the Lorentzian manifolds of GENERAL RELATIVITY.
- As already discussed previously (in Section 10.2.2), Eq. (10.131) is not invariant under arbitrary but only *affine* reparametrizations $\mu = a\lambda + b$. The geodesic equation therefore not only picks out the locally shortest (more precisely: extremal) curves on the manifold, but selects also a particular way to parametrize them (namely a parameter that is proportional to the length of the curve, i.e., an *affine parametrization*).
- The geodesic equation is a second-order differential equation. As such it has a unique solution $x^i(\lambda)$ for any point p of the manifold and tangent vector in $v_p = v_p^i \partial_i \in T_pM$; in coordinates:

$$\left.\begin{aligned} x^i(0) &:= x_p^i \\ \dot{x}^i(0) &:= v_p^i \end{aligned}\right\} \xrightarrow{\text{Eq. (10.131)}} \text{Geodesic } x^i(\lambda) \text{ through } p \text{ in direction } v_p. \tag{10.132}$$

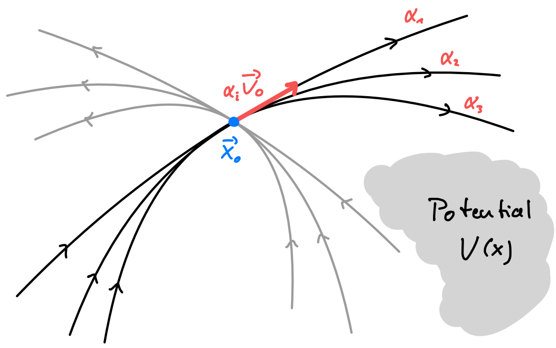
This is reminiscent of classical mechanics where, given some potential $V(\vec{x})$, Newton's law determines a unqiue trajectory for every initial position $\vec{x}_0$ and initial velocity $\vec{v}_0$ of a test particle by solving the second-order differential equation

$$m\vec{x}'' + \nabla V(\vec{x}) = 0\,. \tag{10.133}$$

However, there is a subtle difference between Eq. (10.133) and Eq. (10.131):

Solutions of Newton's equation of motion are *not* invariant under affine reparametrizations in general. That is, if $\vec{x}(t)$ is a solution of Eq. (10.133), the rescaled trajectory $\vec{y}(t) := \vec{x}(\alpha t)$ is no longer a solution (check this!). Note that the effect of the time rescaling α is to scale the initial velocity: $\vec{y}'(0) = \alpha\vec{x}'(0) = \alpha\vec{v}_0$. Physically, this makes sense: If you throw a ball in the *same direction* with *different velocity*, its trajectory will look *different* in a generic potential.

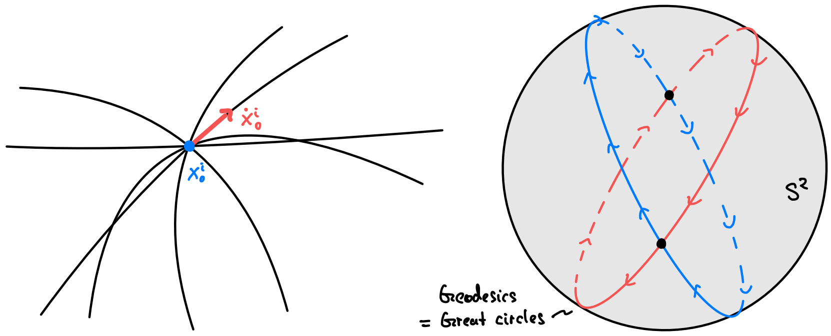
In conclusion, the solutions of Eq. (10.133) form a family of curves through every point, with many different curves going off in the same direction:



Compare this to the geodesic Eq. (10.131):

Given an (affinely parametrized) geodesic $x^i(\lambda)$, which shoots of from $x^i(0)$ in direction $\dot{x}^i(0) = v_p^i$, the reparametrized curve $y^i(\lambda) := x^i(\alpha\lambda)$ is again a solution (check this!). This new curve has again a rescaled tangent vector at p ("initial velocity"), namely $\dot{y}^i(0) = \alpha\dot{x}^i(0) = \alpha v_p^i$. But the two curves $x^i(\lambda)$ and $y^i(\lambda)$ trace out the *same* curve on the manifold, only with a different parametrization ("speed").

The affine reparametrization symmetry of the geodesic equation therefore leads to a *unique* geodesic shooting off in every direction $v_p \in T_pM$ at every point $p \in M$. Rescaling v_p produces the same geodesic, only with a different parametrization (left sketch):



Note that geodesics emanating from a point can meet and cross each other at other points of the manifold (this depends on the curvature, and therefore the metric).

An example is the sphere (right sketch); its geodesics are great circles. At every point of the sphere there is a unique great circle for every direction. But two great circles shooting off in different directions eventually cross again at the antipode of the point where the started from.

- You may wonder: If we know all (unparametrized/projected) geodesics through all points in all directions, do we then know the metric of the manifold? This question is actually of physical significance in GENERAL RELATIVITY, where the geodesics of spacetime correspond to the trajectories of free falling bodies (→ *later*). In the language of GENERAL RELATIVITY, the question then asks whether one can reconstruct the metric of spacetime by observing enough free falling bodies (asteroids, stars, etc.).

  In its strictest sense, the answer to the question is negative. This is easy to see: Consider $\mathbb{R}^2$ and equip this manifold with (1) the Euclidean metric δ_{ij}, and (2) the Minkowski metric η_{ij}. Since both metrics are constant, their Christoffel symbols vanish identically and the solutions of the geodesic Eq. (10.131) are all straight lines for both metrics. On says that the two metrics are ↑ *geodesically equivalent*.

  However, in general it turns out that this is a quite subtle question to answer, see Ref. [136]. Note that one must carefully distinguish between *unparametrized* geodesics (you know only the traces of geodesics on the manifold), and *(affinely) parametrized* geodesics (where you know also the lengths along the traces). Despite the example above, it turns out that *generic* metrics *can* be characterized by their geodesics (even unparametrized ones); i.e., two metrics being geodesically equivalent is not the norm but the exception.

- Imagine you are given a Riemannian manifold and a machine that, input two nearby points on the manifold, spits out the affinely parametrized geodesic through these points (i.e., a curve with "distance ticks" on it). Using this device, you can reconstruct the Levi-Civita connection on the manifold (i.e., you can use it to parallel transport tangent vectors) via a geometric construction known as ↑ *Schild's ladder* [↑ Misner *et al.* [2] (§10.2, pp. 248–249)].

  Fun fact: There is also a science fiction novel called *Schild's Ladder* [137] by the Australian mathematician and Hugo Award winning author GREG EGAN. If you are a fan of hard, mind-bending science fiction à la LEM, ASIMOV and HEINLEIN (and not afraid to encounter concepts from your physics courses in a work of fiction), you might give his novels a try.

- On a Riemannian manifold with a generic metric-compatible connection, that is not necessarily the torsion-free Levi-Civita connection, the coefficients $\Gamma^i{}_{kl}$ in Eq. (10.131) are still the Christoffel symbols (which no longer equal the connection). So the geodesic equation on such a manifold still reads (now with the alternative notation for Christoffel symbols to distinguish them from the connection coefficients):

$$\frac{d^2x^i}{d\lambda^2} + \begin{Bmatrix} i \\ kl \end{Bmatrix} \frac{dx^k}{d\lambda}\frac{dx^l}{d\lambda} = 0\,. \tag{10.134}$$

This equation determines the "shortest lines" (geodesics) on the manifold.

By contrast, the "straightest lines" (autoparallels) are determined by the autoparallel equation Eq. (10.60):

$$\frac{d^2x^i}{d\lambda^2} + \Gamma^i{}_{kl}\frac{dx^k}{d\lambda}\frac{dx^l}{d\lambda} \overset{10.81}{=} \frac{d^2x^i}{d\lambda^2} + \left[\begin{Bmatrix} i \\ kl \end{Bmatrix} - K^i{}_{kl}\right]\frac{dx^k}{d\lambda}\frac{dx^l}{d\lambda} = 0\,. \tag{10.135}$$

Here we used the general form of a metric-compatible connection Eq. (10.81) with the ← *contorsion tensor* $K^i{}_{kl}$. Introducing the symmetric part $K^i{}_{(kl)} \stackrel{\circ}{=} \frac{1}{2}(S_l{}^i{}_k + S_k{}^i{}_l)$ of the contorsion tensor yields (for reference see e.g. [138])

$$\frac{d^2x^i}{d\lambda^2} + \begin{Bmatrix} i \\ kl \end{Bmatrix} \frac{dx^k}{d\lambda}\frac{dx^l}{d\lambda} = K^i{}_{(kl)}\frac{dx^k}{d\lambda}\frac{dx^l}{d\lambda}\,. \tag{10.136}$$

The geodesic equation and the autoparallel equation are therefore equivalent if and only if the symmetric part $K^i{}_{(kl)}$ of the contorsion tensor vanishes (a sufficient, but not necessary, condition is that the torsion $S^i{}_{kl}$ vanishes).

In conclusion, knowing all the geodesics on a manifold only conveys information about the symmetric part of the connection; the *geodesics* know nothing about torsion (but autoparallels do, at least partially). Thus, for a generic metric-compatible connection, there is a difference between "shortest lines" (geodesics) and "straightest lines" (autoparallels).

In GENERAL RELATIVITY, where we only use the torsion-free Levi-Civita connection, we do not have to make this distinction, so that autoparallels and geodesics are the same.

- If the metric $g_{ij}(x)$ is *independent* of a coordinate x^i, Eq. (10.128) implies for the allowed choice $\chi(x) = x/2$

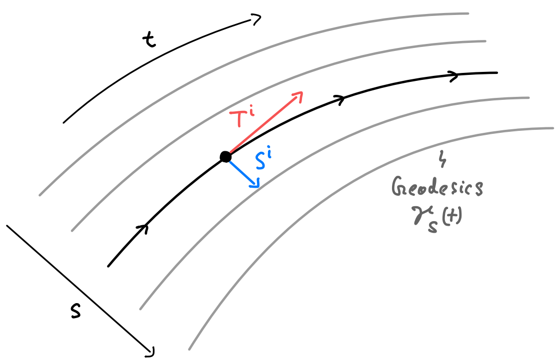
$$p_i := g_{ik}\dot{x}^k = \text{const}\,. \tag{10.137}$$

This "constant of motion" corresponds to the ↓ *cyclic variable* x^i and can be used to simplify the solution of the geodesic equation.

22 | Geodesic deviation:

Details: ➔ Problemset 3

i | ∢ Continuous family of nearby (non-crossing) geodesics $\gamma_s^i(t)$:



Define two vectors fields:

$$T^i := \frac{\partial \gamma_s^i(t)}{\partial t} \quad (\text{“Velocity”}) \quad \text{and} \quad S^i := \frac{\partial \gamma_s^i(t)}{\partial s} \quad (\text{“Deviation”})\,. \tag{10.138}$$

$\triangleleft$ Relative acceleration of nearby geodesics:

$$A^i := \frac{\mathrm{D}^2 S^i}{\mathrm{D}t^2} \overset{10.49}{=} T^n \left(T^m S^i{}_{;m} \right)_{;n} \,. \tag{10.139}$$

The covariant acceleration A^i measures whether two infinitesimally close geodesics “attract” or “repel” each other.

ii | Using the ← *geodesic equation* and the ← *Ricci identity*, one finds:

Eqs. (10.71) and (10.131) →

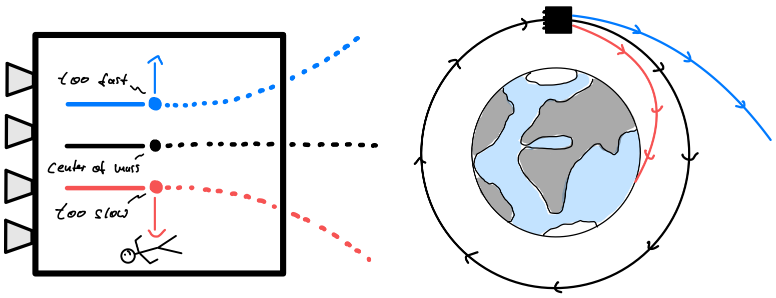
$$\boxed{\frac{\mathrm{D}^2 S^i}{\mathrm{D}t^2} \triangleq R^i{}_{jkl} T^j T^k S^l \qquad ⁂\ \textit{Geodesic deviation equation}} \tag{10.140}$$

Proof: ➔ Problemset 3

(Note that the geodesic equation can be written as $T^k T^i{}_{;k} = 0$.)

→ Curvature makes parallel geodesics attract/repel each other!

iii | But this looks very much like gravity (more precisely: the *tidal effects* of gravity):



(Note that this sketch is a projection of geodesics from *spacetime* to *space*.)

→ Reasonable approach to a *geometric* theory of gravity:

- Free-falling bodies follow *geodesics* in spacetime: → *Chapter 11*
- Masses create *curvature* of spacetime: → *Chapter 12*