↓Lecture 18 [23.04.24]

9.2. General relativity and covariance, background independence

The equivalence principle **EEP** is the foundation of GENERAL RELATIVITY; it motivates both the metrization of gravity (Section 9.4 and ??) and the minimal coupling of matter to gravity (??). However, there are additional principles that are conceptually important to understand and were historically important for the genesis of GENERAL RELATIVITY as well:

10 | Motivation:

- No global inertial systems anymore → More general coordinate charts needed!

  As we have seen, the description of gravity forces us to give up the restriction to formulate physical models within the distinguished family of infinitely extended inertial coordinate systems. Hence we must formulate our physical theories in a way that is valid for arbitrary coordinate charts, and allows for arbitrary coordinate transformations between them.

  Einstein was not satisfied with the distinguished role of inertial systems in SPECIAL RELATIVITY. After all, RELATIVITY is all about the *relativity of states of motion*, i.e., only motion of systems with respect to one another are of physical significance – and no class of states of motion should be distinguished. SPECIAL RELATIVITY clearly does not live up to this rigorous form of relativity as it singles out inertial frames as special. Einstein's ultimate goal was to make all states of motion (including accelerated motion) "equivalent." GENERAL RELATIVITY *does not achieve this goal!* Even in GENERAL RELATIVITY, inertial motion is physically distinct from accelerated motion; the new thing is that mass and energy determine which states of motion *are* inertial.

  We are therefore in the strange (and confusing) situation, that Einstein's original motivation to seek out equations that have "the same form" in all coordinate systems does not achieve its goal, but nevertheless is the correct way forward (see → *next* point). We will also see that "having the same form" means something different in SPECIAL RELATIVITY than in GENERAL RELATIVITY because the former is formulated on a fixed background (Minkowski space) and the latter not (→ *background independence*) – and this changes what it means for two equations to have "the same form." The situation is quite convoluted and we can disentangle it not until the end of this course.

- Chapter 3: *Physics describes relations of geometric entities* ("Coordinates don't exist.")

  → Coordinates should play no role in the formulation of physical models!

  Recall our motivation in Chapter 3 for the introduction of tensor fields: We realized that coordinates are mathematical artifacts that we use to label events in spacetime. The essence of physical laws should clearly be independent of the labeling scheme we choose. Thus we should strive for a formulation of physical models (which, hopefully, capture physical laws) that is independent of coordinates, or at least makes it manifest that physical predictions do not depend on the choice of coordinates.

  Note that this argument is very different from Einstein's hope to extend the principle of special relativity **SR** by "equalizing" more states of motion. The argument is way more fundamental, has nothing to say about states of motion, and, in some sense, is almost tautological. It's only physical content is the rather uncontroversial statement that "coordinates do not exist as physical entities."

11 | This motivates the following definition:

★ **Definition 1: General covariance GC (*Coordinates don't exist*)**

An equation is said to be ⁂ *generally covariant* if it is forminvariant under arbitrary (differentiable) coordinate transformations.

→ Tensor equations are automatically generally covariant.

Generally covariant equations have an alternative ⁂ *coordinate-free* formulation in terms of geometric objects on a manifold (↑ *differential forms*):

Examples:

- ⊲ Two vector fields $A = A^\mu \partial_\mu$ and $B = B^\mu \partial_\mu$:

$$\underbrace{\left\{\begin{array}{c|l} \phi_A = \phi_B & \bar\phi_A = \frac{\partial \bar x^0}{\partial x^0}\phi_A + \frac{\partial \bar x^0}{\partial x^i}A^i \\ \vec{A} = \vec{B} & \bar A^i = \dots \\ & \bar\phi_B = \dots \\ & \bar B^i = \dots \end{array}\right\}}_{\text{Generally covariant}} \Leftrightarrow \underbrace{A^\mu = B^\mu}_{\substack{\textit{Manifestly}\\ \text{(generally)}\\ \text{covariant}}} \Leftrightarrow \underbrace{A = B}_{\substack{\text{Coordinate}\\ \text{free}}} \tag{9.10}$$

  - While mathematicians often prefer the coordinate-free notation, in physics, the coordinate-dependent, manifestly covariant notation is more widespread. This has to do with how physics is done: While coordinates do not exist *a priori*, physicists typically make them exist in their labs because measurements always use some form of reference system. The generally covariant equations are more useful in that regard because they can be specialized to any coordinate system most convenient for an experiment.

    → In this course we will only use the manifestly covariant notation.
  - To decided whether an equation remains form invariant under arbitrary coordinate transformations, you must first know how the elementary fields of the equation transform. This is why the *non*-manifest notation is so cumbersome: In addition to the equation(s), you must figure out (or specify) how the different fields transform. (Recall the non-tensorial form of the Maxwell equations and how cumbersome it was to check their Lorentz covariance [see Eq. (6.34)]!)

    This makes the benefit of the *manifest* notation clear: First, *by convention*, the tensor notation A^μ implies that the transformation of the field is $\bar A^\mu = \frac{\partial \bar x^\mu}{\partial x^\nu} A^\nu$, and second, because of the rules of tensor calculus, checking the general covariance of a (valid) tensor equation is trivial.
- ⊲ Inhomogeneous Maxwell equations on arbitrary spacetime (??):

$$\underbrace{F^{\mu\nu}{}_{;\nu} = -\tfrac{4\pi}{c} J^\mu}_{\text{Manifestly covariant}} \Leftrightarrow \underbrace{d(\star F) = \star J}_{\text{Coordinate-free}} \tag{9.11}$$

  Remember that $;\nu$ denotes the ← *covariant derivative* Eq. (3.79) which implicitly depends on the metric of spacetime. In the coordinate-free notation, the metric is hidden in the definition of the ↑ *Hodge star operator* $\star$.

12 | We can now use this definition to formulate our physical insight that the equations that describe physical laws must not single out specific coordinate systems:

In Einstein's words [20]:

> *Die Gesetze der Physik müssen so beschaffen sein, daß sie in bezug auf beliebig bewegte Bezugssysteme gelten. (p. 772)*
>
> *Die allgemeinen Naturgesetze sind durch Gleichungen auszudrücken, die für alle Koordinatensysteme gelten, d.h. die beliebigen Substitutionen gegenüber kovariant (allgemein kovariant) sind. (p. 776)*

→

§ Postulate 4: General relativity GR

Models of laws of nature must take the same form in *all* coordinate systems; i.e., they must be expressed in terms of generally covariant equations.

Here the "must take the form" means that it must be *possible* to formulate them in a coordinate-independent way; if this were not the case, the theory (and its prediction) would implicitly depend on (and single out) a specific coordinate system. Note that there is nothing wrong in formulating such a theory in a way that is not *generally covariant*.

For example, Maxwell equations in their conventional (non-tensorial) form are not generally covariant, they are only Lorentz covariant. This is not a problem, though, because these equations are just a specialization of Eq. (9.11) to a particular class of coordinate systems (namely inertial systems). If you (naïvely) apply these specialized equations in a non-inertial frame (such as a laboratory on the surface of earth!), you can get seemingly paradoxical results (➔ Problemset 3 and Ref. [113]).

13 | What is the physical content of GR?

GR, while being important for the formulation of physical models in general and being strictly satisfied in GENERAL RELATIVITY, is neither specific nor fundamental to and for GENERAL RELATIVITY. For example, the Maxwell equations in the manifestly covariant form of Eq. (9.11) satisfy the GR on the fixed background of Minkowski space and have nothing to do with GENERAL RELATIVITY.

→

The principle of general relativity GR has (almost) no physical content.

The "almost" referes to the fact that the principle asserts that there are no distinguished coordinate systems that exist as physically independent structures.

- The relativity postulate GR, and its mathematical manifestation as general covariance GC have been criticized already in 1917 by KRETSCHMANN [114]:

  > *[Man] vergegenwärtigt sich, daß alle physikalischen Beobachtungen letzten Endes in der Feststellung rein topologischer Beziehungen ("Koinzidenzen") zwischen räumlich-zeitlichen Wahrnehmungsgegenständen besteht und daher durch sie unmittelbar kein Koordinatensystem vor irgend einem anderen bevorrechtigt ist, so wird man zu dem Schlusse gezwungen, daß* jede physikalische Theorie *ohne Änderung ihres–beliebigen–durch Beobachtungen prüfbaren Inhaltes mittels einer rein mathematischen und mit*

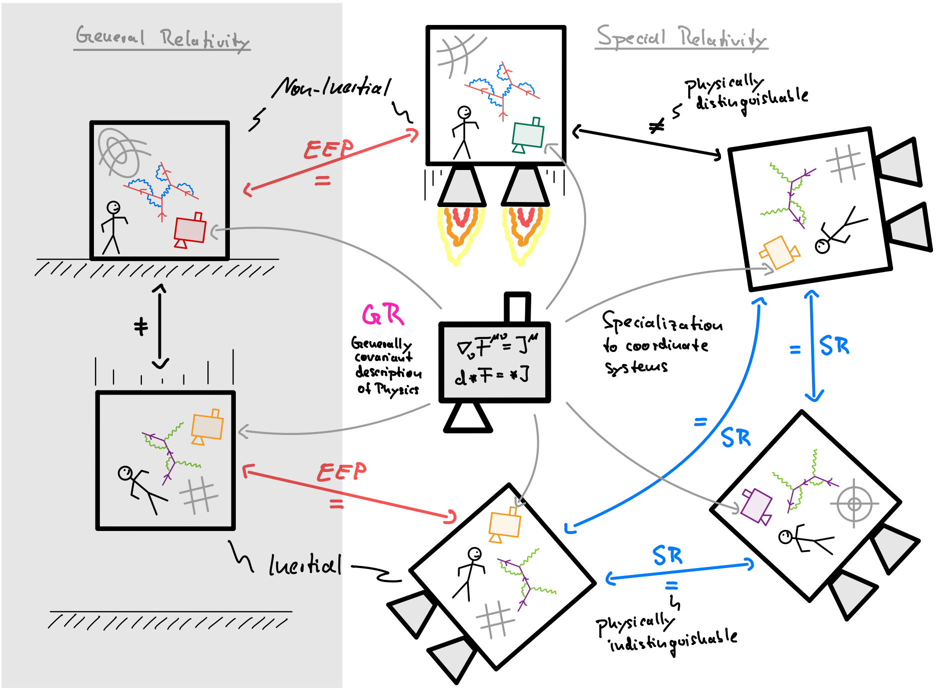
> *höchstens mathematischen Schwierigkeiten verbundenen Umformung der sie darstellenden Gleichungen mit jedem beliebigen–auch dem allgemeinsten–Relativitätspostulate in Einklang gebracht werden kann.*

- To drive the point home: One can also formulate good old non-relativistic Newtonian mechanics in a generally covariant form (it's quite ugly, though)! See the original literature [115] and MISNER *et al.* [2] (Box 12.4 and §12.5):

  > *Any physical theory originally written in a special coordinate system can be recast in geometric, coordinate-free language. Newtonian theory is a good example [..]. Hence, as a sieve for separating viable theories from nonviable theories, the principle of general covariance is useless.*

- For a detailed account on the role general covariance plays in GENERAL RELATIVITY (and historically played in its inception), see Ref. [116].

14 | We can summarize the relation of SR, EEP, and GR as follows:



- Both SR and EEP make claims about the equivalence (indistinguishability) of certain states of motion. These are physical claims about reality that can be assessed by experiments. Note that to check whether they are false or true you do not even know how to work with mathematical equations. It's a simple matter of collecting the results of experiments (recall the ← *Michelson-Morley experiment*). It is this physical content that makes SR and EEP the foundations of SPECIAL RELATIVITY and GENERAL RELATIVITY, respectively.
- By contrast, GR makes no such claims about reality. GR does *not* claim that all states of motion are indistinguishable (they are not, even in GENERAL RELATIVITY you can tell local inertial frames and accelerated frames apart); the principle only claims that all fundamental theories of physics should have a formulation that can be applied by all possible observers. GR is therefore more a statement about physical models than about reality.
- The sketch makes it clear that the EEP is actually more similar to the SR (in the role it plays for GENERAL RELATIVITY) than the GR is. In that sense "principle of general relativity" is kind of a misnomer.

15 | There is another important concept that (contrary to GC/GR) distinguishes GENERAL RELATIVITY from other theories and must itself be distinguished from GC/GR:

★ **Definition 2: Background independence** BI (*No prior geometry*)

Physical models that do not contain the geometry of spacetime as an absolute element are called ⁂ *background independent*. This implies that the geometry of spacetime emerges dynamically as *solutions* of the theory.

(Counter-)examples:

- ✓ GENERAL RELATIVITY is (and historically was the first example of) a background independent theory (→ *below*):

$$S_{\text{Einstein-Hilbert}}[g] = \frac{c^4}{16\pi G}\int d^4x\,\sqrt{g}\,R \tag{9.12}$$

  Here R is the → *Ricci scalar* that depends in a complicated way on the metric tensor field $g_{\mu\nu}(x)$. $\sqrt{g}$ is short for $\sqrt{|\det[g_{\mu\nu}(x)]|}$ (Minkowski metric: $\sqrt{\eta}=1$).

- ✗ Maxwell theory is *not* background independent (recall Eq. (6.56)):

$$S_{\text{Maxwell}}[A] = \int d^4x\,\sqrt{g}\left(-\frac{1}{16\pi}g_{\mu\alpha}g_{\nu\beta}F^{\alpha\beta}F^{\mu\nu}\right) \tag{9.13}$$

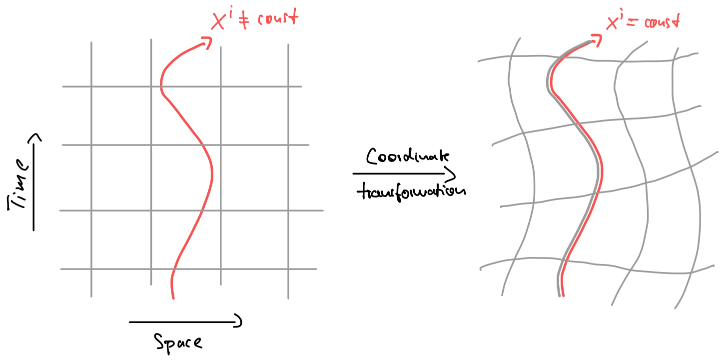
  with field-strength tensor $F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu$.

  - Note that you do *not* extremize this action wrt. the metric g; the metric (e.g. Minkowski metric $g=\eta$) is given as a *parameter* (absolute element) of the theory. Hence it plays the role of a static background.
  - Note also that the Maxwell equations (and their Lagrangian) *are* generally covariant: they are tensorial expressions that descibe geometric objects on a manifold. That does not prevent them to have a tensor field (the metric) as an absolute element.

16 | Beware:

GENERAL RELATIVITY is most likely the first (and possibly last) generally covariant and background-independent theory that you will encounter in your university courses. Thus it is important to mention a peculiarity that, if ignored, can lead to lots of confusion when studying such theories:

i | ⊲ Trajectory of particle in spacetime:

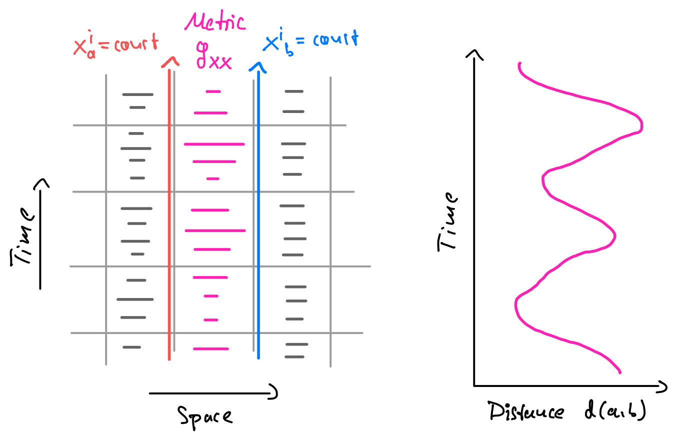


Because of general covariance (GC) there is always a coordinate system in which the (spatial) coordinates of an object are constant in time.

→ One cannot infer from coordinates whether an object is moving!

(There is no absolute notion of "motion" in GENERAL RELATIVITY; the only motion that makes sense is motion wrt. some reference object, → *next step*.)

ii | ⊲ Two objects at rest in some coordinate system:



In a background-independent theory (**BI**) the metric is a dynamic degree of freedom. Therefore two objects a and b can have *constant coordinates* in space while their distance varies over time! Note that the coordinates are completely independent of the metric in general.

→ One cannot infer from coordinates whether distances of objects change!

We can sum this up as follows:

> Coordinates have no physical meaning in GENERAL RELATIVITY; they are simply (arbitrary) labels of events.

- Please make sure you grasp this statement fully (we will see explicit examples later when we understand GENERAL RELATIVITY better):

  If I tell you in SPECIAL RELATIVITY that (in some inertial frame) two test particles have constant spatial coordinates x^i and y^i, you immediately know their relative velocity and distance: $\vec{v}_{\text{rel}} = \dot{\vec{x}} - \dot{\vec{y}} = \vec{0}$ and $\Delta l^2 = -(x^\mu - y^\nu)^2 = |\vec{x} - \vec{y}|^2$.

  By contrast, if I tell you in GENERAL RELATIVITY that two test particles have (in some coordinate system) constant spatial coordinates x^i and y^i, this tells you *nothing* about their distance; not even whether it is constant or varies in time! This information is hidden in the values of the metric field, not in the coordinates.

- This is important, for example, when studying the effects of gravitational waves (→ *later*).

17 | Summary:

> - Every reasonable fundamental theory has a generally covariant formulation.
> - A generally covariant theory does not need to be background independent.
> - GENERAL RELATIVITY is background independent and generally covariant.

- We will return to the question of general covariance, background independence (and diffeomorphism invariance) → *later* when we know more about GENERAL RELATIVITY. At this point it is only important that you know the conceptual difference between the terms *general* *covariance* GC and *background independence* BI and the requirement of the *principle of general relativity* GR.

- One sometimes hears that general covariance GC is a distinguishing feature of GENERAL RELATIVITY (as explained above, it is not). Sometimes even the very concepts of general covariance GC and background independence BI are confused. This confusion is partially rooted in history because Einstein himself didn't separate the two concepts clearly. MISNER *et al.* explain [2] (p. 431):

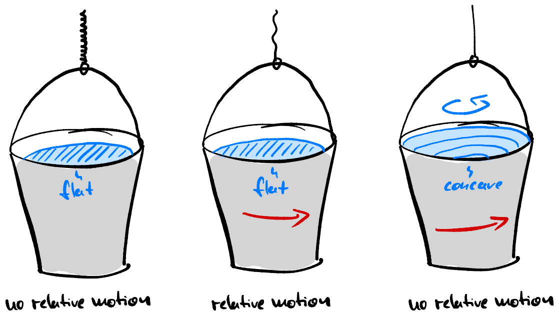
  > *Mathematics was not sufficiently refined in 1917 to cleave apart the demands for "no prior geometry" and for a "geometric, coordinate-independent formulation of physics." Einstein described both demands by a single phrase, "general covariance." The "no-prior-geometry" demand actually fathered general relativity, but by doing so anonymously, disguised as "general covariance," it also fathered half a century of confusion.*

- For more details on the relation between the concepts of background independence, general covariance, and diffeomorphism invariance see Ref. [117] (and references therein).

9.3. Mach's principle (an its failure in GENERAL RELATIVITY)

Mach's principle is not a logical postulate of GENERAL RELATIVITY and mostly of historical (and perhaps philosophical) importance. However, it is also conceptually interesting because at a fist glance once might conclude (as Einstein did), that GENERAL RELATIVITY actually satisfies the principle. It is rather subtle (and instructive) why this is not so:

18 | Recall ← *Newton's bucket*:



Question: Rotation with respect to *what* determines the shape of the water surface?

NEWTON: Absoute space!

Note that the experiment already suggests that this "absolute space" must have certain symmetries since the experiment cannot distinguish specific points nor specific states of uniform motion. SPECIAL RELATIVITY tells us that the correct symmetry group of spacetime is the Poincaré group. So from our modern perspective, Newton's answer must be read as follows: *The experiment demonstrates the independent existence of an entity which determines the local inertial systems.* We may call this entity "spacetime."

19 | The austrian physicist ERNST MACH fervently disagreed with Newton [118]:

> *Der Versuch Newton's mit dem rotirenden Wassergefäss lehrt nur, dass die Relativdrehung des Wassers gegen die Gefässwände keine merklichen Centrifugalkräfte weckt, dass dieselben aber durch die Relativdrehung gegen die Masse der Erde und die übrigen Himmelskörper geweckt werden. Niemand kann sagen, wie der Versuch verlaufen würde, wenn die Gefässwände immer dicker und massiger, zuletzt mehrere Meilen dick würden. Es liegt nur der eine Versuch*

vor, und wir haben denselben mit den übrigen uns bekannten Thatsachen, nicht aber mit unsern willkürlichen Dichtungen in Einklang zu bringen.

Mach denied Newton's notion of an independent entity responsible for inertia and proposed that the large-scale structure of matter in the cosmos determines the local inertial systems instead (↑ *relationalism*):

§ Principle: Mach's principle MP

Local inertial frames are determined by the cosmic motion and distribution of matter.

- Mach never formulated his principle precisely, which leaves room for interpretation and personal taste. This is why there are various readings of MP in the literature, not all equivalent. The above phrasing is a rather strict version of the principle.
- Here is an alternative way to illustrate the point by Steven Weinberg [119] (p. 17):

  > *There is a simple experiment that anyone can perform on a starry night, to clarify the issues raised by Mach's principle.*
  >
  > *First stand still, and let your arms hang loose at your sides. Observe that the stars are more or less unmoving, and that your arms hang more or less straight down. Then pirouette. The stars will seem to rotate around the zenith, and at the same time your arms will be drawn upward by centrifugal force. It would surely be a remarkable coincidence if the inertial frame, in which your arms hung freely, just happened to be the reference frame in which typical stars are at rest, unless there were some interaction between the stars and you that determined your inertial frame.*

  Put this way, the situation is quite puzzling indeed and Mach's principle doesn't seem far fetched at all.
- Einstein was responsible for coining the term "Mach's principle" and was influenced by it during his construction of general relativity. At first, he believed that in his new theory of gravity the principle was indeed satisfied. He writes in a letter to Mach in 1913 [120]:

  > *Dieser Tage haben Sie wohl meine neue Arbeit über Relativität und Gravitation erhalten, die nach unendlicher Mühe und quälendem Zweifel nun endlich fertig geworden ist. Nächstes Jahr bei der Sonnenfinsternis soll sich zeigen, ob die Lichtstrahlen an der Sonne gekrümmt werden, ob m. a. W. die zugrunde gelegte fundamentale Annahme von der Aequivalenz von Beschleunigung des Bezugssystems einerseits und Schwerefeld andererseits wirklich zutrifft.*
  >
  > *Wenn ja, so erfahren Ihre genialen Untersuchungen über die Grundlagen der Mechanik – Planck's ungerechtfertigter Kritik zum Trotz – eine glänzende Bestätigung. Denn es ergibt sich mit Notwendigkeit, dass die Trägheit in einer Art Wechselwirkung der Körper ihren Ursprung hat, ganz im Sinne Ihrer Überlegungen zum Newton'schen Eimer-Versuch.*

  (You may wonder how Einstein could write this letter in 1913 when he finalized general relativity in November of 1915. Einstein refers to his paper with Marcel Grossmann published in 1913 [121] in which they established the "Entwurftheorie", a precursor of general relativity that already included most of the pieces needed [but not yet the correct field equations].)

20 | So here is the case Mach (& early Einstein) vs. Newton:

- NEWTON:

  Space exists as an independent entity and determines locally which frames are inertial.

- MACH:

  Space emerges from the relations between matter and does not exist independently. Hence the distribution of matter in the universe completely determines the local inertial systems.

Who is right according to GENERAL RELATIVITY?

21 | Answer: Both …(in a sense, though Newton is more correct)

- NEWTON's conclusion was correct: Because of locality (constancy of the speed of light) the matter distribution of the cosmos (the fixed stars) cannot immediately influence the local inertial frame; there must be a mediator, some "background" that is here right now. GENERAL RELATIVITY tells us what this is: the metric tensor field that determines the geometry of spacetime.
- MACH was right insofar as it is indeed not a coincidence that the local inertial frame on earth is at rest with respect to the fixed stars. There *is* a relation, although not a direct and immediate one. GENERAL RELATIVITY tells us that the large-scale distribution of matter (and energy) in the universe determines the (large-scale) metric of spacetime, which, in turn, determines the local inertial systems everywhere. But there is a hitch: the metric is not uniquely determined by the mass distribution. Thus the metric (and therefore spacetime) carries independent degrees of freedom. There is more than matter in the world, spacetime is a real entity!

Notes:

- Today we know that there are solutions of the Einstein field equations (e.g. the ↑ *Gödel universe* [122]) that violate Mach's principle explicitly [123].
- MACH, in his critique of NEWTON's bucket experiment, asked (rhetorically) what would happen if the walls of the bucket would become very thick and massive. His point was that it is not excluded that at some point the rotation of the bucket *would* influence the shape of the water. GENERAL RELATIVITY tells us that this is so indeed, because a very massive bucket affects the geometry of spacetime. This is known as the → *Lense-Thirring effect* [124, 125] (also know as ↑ *frame-dagging*) and has been experimentally confirmed (not with a massive bucket, of course, but with earth) [126, 127]. However, for the reasons explained above, this effect does not make GENERAL RELATIVITY comply with Mach's principle in the strict sense.
- Because of the many different versions of MP floating around, for some the case is still not closed (at least for philosophers of science, it seems). For doing physics with GENERAL RELATIVITY, MP is irrelevant.
- MACH advocated a relational view of space(time): Only relations between the degrees of freedom of matter are observable. There is no independent meaning of, say, an electron being *here now*. It is interesting to realize that this relational view might very well be true (and in accordance with GENERAL RELATIVITY) if one accepts that the metric field is just another collection of degrees of freedom which can be in relations (coincide or interact) with other degrees of freedom. For example, an electron being *here now* might simply mean that an excitation of the field that describes the electron coincides/interacts with a particular degee of freedom of the metric field.

22 | Conclusion:

The controversy about the MP essentially boils down to the question whether spacetime has independent degrees of freedom (and therefore exists in a physical sense):

In its strict version, the MP denies spacetime this independent role. By contrast, GENERAL RELATIVITY grants spacetime independent degrees of freedom because the → *Einstein field equations* only constrain the → *Einstein tensor* but not the metric directly (→ *Gravitational waves*):

> GENERAL RELATIVITY *violates* Mach's principle MP because matter *influences* the geometry of spacetime but does not *determine* it uniquely.

This situation is exemplified by gravitational waves: When in 2015 the interferometers of LIGO detected a gravitational wave passing earth, the spacetime geometry in our vicinity changed by a very tiny bit. However, the mass distribution in the vicinity of earth didn't change at all. So while the geometry of spacetime certainly is *influenced* by earths mass, it is not uniquely *determined* by it. LIGO therefore measured directly the dynamics of the degrees of freedom the existence of which MP denies.

9.4. Overview and Outline

Now that we know the conceptual starting point of GENERAL RELATIVITY, and argued that more general spacetimes than flat Minkowski space are needed to accomodate gravity, we can reveal the gist of GENERAL RELATIVITY and sketch the plan for the remainder of this course:

¡! You are not required to fully grasp the *how* and *why* of the following statements. Understanding the details is the objective of this course. However, I think that it is useful to start off with a rough picture of what we want to accomplish because otherwise one is easily swamped by the details along the way.

GENERAL RELATIVITY in a Nutshell

- Ontology:

$$\boxed{\begin{aligned}\text{Spacetime} &\equiv \text{4D differentiable manifold } M\\ \text{Gravitational field} &\equiv \text{pseudo-Riemannian metric } g \text{ with signature } (1,3)\end{aligned}}$$

  → Spacetime is a ← *4D Lorentzian manifold*

  - Note that we only fix the dimensionality of M (and thereby its *local* topology) but not its *global* topology (i.e., whether it is simply $\mathbb{R}^4$, a sphere, a torus, or something even more fancy). Thus, for example, GENERAL RELATIVITY makes no a priori statement about the finiteness of the universe. (Asking about the *local* topology is like asking where space and time come from—and GENERAL RELATIVITY is silent about that. A reasonable theory of quantum gravity must address this question!)
  - At this stage it is sufficient to interpet the points $E \in M$ of the manifold as points in spacetime and therefore as (equivalence classes of) events. However, we will see that this interpretation is problematic (→ *Hole argument*) because of the diffeomorphism invariance of GENERAL RELATIVITY. It is thus questionable whether points of the manifold (and thereby the manifold itself) can be associated to any existing entity. An entity that certainly does exist, however, is the metric field.

- <u>Equivalence principle:</u>

  The EEP is built right into the mathematical framework of GENERAL RELATIVITY:

  For every point with coordinates y, there exists a coordinate transformation φ_y with $\bar{x} = \varphi_y(x)$ such that:

$$\bar{g}^{\mu\nu}(\bar{x}) = \frac{\partial \bar{x}^\mu}{\partial x^\alpha}\frac{\partial \bar{x}^\nu}{\partial x^\beta} g^{\alpha\beta}(x) \overset{x\approx y}{\approx} \eta^{\mu\nu} \quad \text{and} \quad \bar{\partial}_\rho \bar{g}^{\mu\nu}(\bar{x}) \overset{x\approx y}{\approx} 0 \tag{9.14}$$

  with Minkowski metric

$$\eta^{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}_{\mu\nu} . \tag{9.15}$$

  → *Locally* inertial coordinates ✓(➔ Problemset 2)

$$\boxed{\text{The EEP motivates the metrization of gravity.}} \tag{9.16}$$

  - ¡! In the presence of gravity, there is *no* coordinate transformation that brings the metric into Minkowski form *everywhere* on the spacetime manifold. Conversely, if this *is* possible, spacetime is flat Minkowski space and you were doing SPECIAL RELATIVITY all along (perhaps in curvilinear coordinates).
  - Note that metrization of gravity is not a *mathematical corollary* of the EEP (the latter is a physical principle, not a rigorous mathematical statement). However, the EEP is most naturally incorporated into a mathematical framework where gravity is described by a metric because the gist of the EEP is that all (local) physical phenomena are affected by gravity in the same way. This is exactly what happens if gravity is identified with the geometry of spacetime!
  - At every point, the basis $\{\bar{\partial}_\rho\}$ of the tangent space forms a so called ⁂ *local Lorentz frame*. You can choose such a basis for all points of spacetime. However, in general there is no coordinate system that induces this basis everywhere; you have to use multiple charts to patch together spacetime.

- <u>Important fields:</u>

  All degrees of freedom (some gauge, some physical) of GENERAL RELATIVITY are stored in the metric tensor field. From the metric, one can then derive other fields that play important roles in the formulation of the theory:

$$\begin{tikzcd}
 & & & \overbrace{R_{\mu\nu}}^{\text{Ricci tensor}} \arrow[d, "\mathrm{Tr}"] \arrow[dr, "\text{Ch. ??}"] & \\
\underbrace{\overbrace{g_{\mu\nu}(x)}^{\text{Metric tensor}}}_{\widehat{=}\ \text{Gravitational potential}} \arrow[r, "\partial g", "\text{Sec. ??}"'] & \underbrace{\overbrace{\Gamma^\lambda{}_{\mu\nu}}^{\text{Connection (no tensor)}}}_{\widehat{=}\ \text{Gravitational field strenght}} \arrow[r, "{\partial\Gamma, \Gamma^2}", "\text{Sec. ??}"'] & \underbrace{R_{\rho\sigma\mu\nu}}_{\text{Curvature tensor}} \arrow[ur, "\mathrm{Tr}"] & & \underbrace{G_{\mu\nu}}_{\text{Einstein tensor}} \\
 & & & \underbrace{R}_{\text{Ricci scalar}} \arrow[ur, "\text{Ch. ??}"'] &
\end{tikzcd}$$

Note that the Einstein tensor is non-linear in the metric an contains (up to) second-order derivatives.

- Einstein field equations (EFE): (here without cosmological constant)

  The centerpiece of GENERAL RELATIVITY is a tensorial partial differential equation that determines the metric tensor field in dependence of the energy momentum tensor of matter:

$$\boxed{\text{Einstein tensor} \rightarrow \underbrace{G_{\mu\nu}}_{\text{"Geometry"}} = \kappa \underbrace{T_{\mu\nu}}_{\text{"Matter"}} \leftarrow \text{Energy-momentum tensor}} \tag{9.17}$$

  ¡! "Matter" refers here to all degrees of freedom that carry energy and/or momentum. This includes bodies with rest mass but also electromagnetic radiation etc.

  ⊲ Eq. (9.17): Non-linear, second-order PDE for $g_{\mu\nu}$:

  GENERAL RELATIVITY describes the geometry of space as a dynamical field that evolves "in time" according to a highly nontrivial PDE: →

$$\boxed{\text{GENERAL RELATIVITY} = ⁂\ \textit{Geometrodynamics}} \tag{9.18}$$

  The nonlinearity makes Eq. (9.17) hard to solve, even in vacuum were the right-hand side vanishes.

  - Spacetime geometry is dynamical → Background independence ✓
  - Tensor equation → General covariance (no preferred coordinate system) ✓
  - *Mass distribution* determines *metric* determines *local inertial frames*
    However: Fixing $G_{\mu\nu}$ leaves some degrees of freedom of $g_{\mu\nu}$ unconstrained!
    → Boundary conditions required for unique solution
    → Mach's principle is not satisfied (but partially survives in spirit) ✗/✓
  - Recall Eq. (8.10) and our discussion that followed (also ➔ Problemset 1). Eq. (9.17) is structurally similar but fixes the problem of linearity because the Einstein tensor is a non-linear function of the metric.

- Physics with gravity:

  Once gravity is described by the metric, one must generalize the other relativistic theories (mechanics, electrodynamics, ...) into a generally covariant form that couples to the metric. This generalization is a priori not unique because matter can couple in various ways to the fields derived from the metric.

  However, the EEP severely restricts the couplings that are allowed and leads to a "recipe" how the Lorentz covariant equations of SPECIAL RELATIVITY must be rewritten to match the principles of GENERAL RELATIVITY (→ *"Comma-Goes-to-Semicolon Rule", Minimal coupling*):

  > The GR demands physical theories to be specified by *tensor equations*.
  > The EEP restricts the possible couplings of matter and metric.

  →

  - Mechanics: (with $u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$ the 4-velocity)

$$m \underbrace{\frac{\mathrm{D}u^\mu}{\mathrm{D}\tau}}_{\substack{\text{Absolute}\\ \text{derivative}}} = m \underbrace{\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2}}_{\text{4-accel.}} + \underbrace{m\,\Gamma^\mu{}_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}}_{\text{New!}} = \underbrace{K^\mu}_{\text{4-force}} \tag{9.19}$$

→ Free particle: $K^\mu = 0$ → Geodesic equation ("straight lines" in spacetime)

- Electrodynamics: ⊲ Inhomogeneous Maxwell equations: (cf. Eq. (6.50))

$$\overbrace{F^{\mu\nu}{}_{,\nu} = -\frac{4\pi}{c} j^\mu}^{\text{SPECIAL RELATIVITY}} \quad \to \quad \overbrace{\underbrace{F^{\mu\nu}{}_{;\nu}}_{\text{Covariant derivative}} = -\frac{4\pi}{c} J^\mu}^{\text{GENERAL RELATIVITY}} \tag{9.20}$$

The covariant derivative contains the connection Γ and therefore the metric g. This is how the electromagnetic field is affected by the gravitational field (e.g., bent in the vicinity of heavy masses). Conversely, the EM field gives rise to the energy momentum tensor $T^{\mu\nu}_{\text{em}}$ [← *Eq.* (6.110)] and thereby contributes to the right-hand side of Eq. (9.17).

→ Energy-momentum tensor $T_{\mu\nu}$ is dynamical

A generally relativistic theory of matter, interacting with and via gravity, is then described by a *coupled, non-linear, higher-order system of partial differential equations* (where $T_{\mu\nu}$ depends on the dynamical variables of the matter theory).

→ Hard to solve in general → Approximations needed!

Outline of this course

Here is our approach for this course to study these various aspects of GENERAL RELATIVITY:

- Step 1 (Section 9.4): *How to describe non-Euclidean manifolds mathematically?*

  In Chapter 3 we introduced the concept of differentiable manifolds and introduced the concept of a (pseudo-)Riemannian metric to measure lengths of curves on the manifold. To formulate GENERAL RELATIVITY mathematically, we need to revisit and extend this toolbox of tensor calculus.

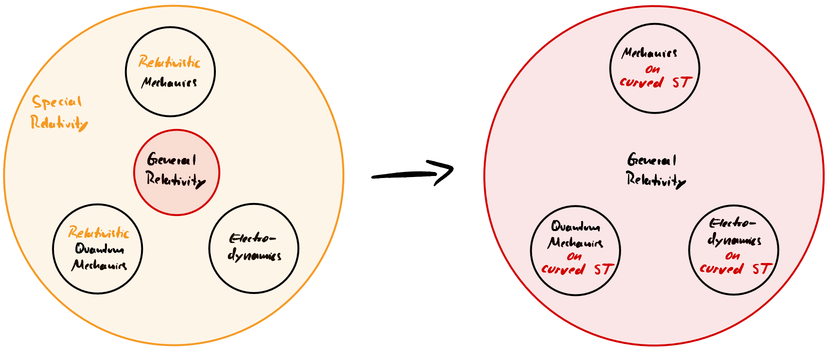
  In particular, we will study two (at first independent) structures that can be put on a differentiable manifold:

  → *Affine connection* → Determines parallel transport, straight lines, and curvature
  ← *Riemannian metric* → Determines lengths, shortest lines, and angles

  As it turns out, when you are given a Riemannian metric, there is a unique way to construct an affine connection. This means that once you are given a spacetime manifold with a (pseudo-)Riemannian (Lorentzian) metric, all concepts in the list above are well-defined. This is then the framework we will use: The spacetime of GENERAL RELATIVITY is a Lorentzian manifold and the degrees of freedom (field) of the theory is the Lorentzian metric itself.

- Step 2 (??): *How to formulate relativistic theories on non-Euclidean spacetimes?*

  In the first part of this course, we first established the tenets of SPECIAL RELATIVITY (Lorentz symmetry) and then incorporated them successively in known theories of physics (point mechanics in Chapter 5, electrodynamics in Chapter 6, quantum mechanics in Chapter 7). Now that we established the tenets of GENERAL RELATIVITY (the spacetime metric is not necessarily the Minkowski metric but an arbitrary Lorentzian metric), we must again reformulate our theories to comply with this new insight. Recall p. 16 in the introduction:



This will lead us to generally covariant formulations of relativistic mechanics in ?? and electrodynamics in ?? (we will skip quantum mechanics this time, but this is also possible). Thanks to the combination of **EEP** and **GR** (and tensor calculus), the recipe to go from the equations of SPECIAL RELATIVITY to that of GENERAL RELATIVITY will be very simple.

- Step 3 (??): *How to determine the geometry of spacetime dynamically?*

  Up until this point we simply declared that the metric of spacetime is an arbitrary Lorentzian metric and studied the effects on physics *given such a metric*. The core idea of GENERAL RELATIVITY (and maybe *the* most important insight of Albert Einstein) was that this metric was not part of the laws of nature but just another degree of freedom that had to be dynamically determined. This means that there is no a priori geometry of spacetime, a principle known as ← *background independence*. The equations that dynamically determine the metric are the → *Einstein field equations* Eq. (9.17); they are the centerpiece of GENERAL RELATIVITY and determine the geometry of spacetime, given the distribution of mass and energy and some boundary conditions. We will derive these equations via an action principle from a Lagrangian.

  Together with Step 2, this completes the framework of GENERAL RELATIVITY.

- Step 4 (??): *What does* GENERAL RELATIVITY *predict?*

  If we combine the results of Step 2 and Step 3 we obtain a self-contained, background independen framework to describe physics: Matter determines the geometry of spacetime (Step 3) and, conversely, this geometry determines how matter evolves (Step 2). This interplay makes for beautiful but mathematically complicated models. Thus, to study the predictions of GENERAL RELATIVITY, we typically resort to simplified approaches:

  - Consider a static, inhomogeneous distribution of large masses (e.g. the sun). Using the Einstein field equations from Step 3 (and reasonable boundary conditions), calculate the geometry of spacetime induced by this distribution. Then use the results of Step 2 to determine the evolution of small test particles on this curved spacetime (without taking their backaction on spacetime into account). This approach leads to a variety of phenomena, e.g., the slowing down of clocks close to large masses (??), the perihelion precession of Mercury (??), the bending of light (??), etc.
  - Consider the Einstein field equations in vacuum, i.e., without any matter (or energy). Because the EFEs are non-linear (recall Section 8.2) and the geometry of spacetime is not uniquely determined by the distribution of mass and energy, this situation is not as boring and trivial as it sounds (its actually very complicated). But even in the weak field limit (where one drops the self-interactions) one finds something interesting: gravitational waves (??).
  - Consider an idealized universe that is homogeneously filled with matter and energy (and potentially dark matter and dark energy). If one calculates the solutions of the EFEs in such a scenario, one obtains the (approximate) spacetime geometry of the whole universe. This leads into the field of ↑ *relativistic cosmology* and to the current standard model of cosmology, known as "ΛCDM". This is where one finds the possiblity of an expanding universe and its origin, the Big Bang; this is also where the cosmological constant becomes important (??).