↓Lecture 17 [16.04.24]

10 | Yet another problem:

Besides the formal complications encountered above, there are also less formal yet fundamental lines of reasoning that suggest that the phenomenon of gravitation and the premises of SPECIAL RELATIVITY are incompatible:

i | Experimental facts:

- Gravity cannot be shielded.

  Contrary to all other forces (which have negative charges), there is no negative mass.

- Gravity is typically *inhomogeneous*.

  In a gravitationally homogeneous universe there are no planets and we wouldn't exist.

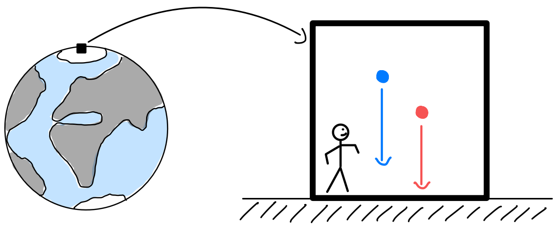
- In free fall, gravity is exactly countered by the inertial force.

  We will discuss this in more detail later (→ *equivalence principle*).

ii | For the machinery of SPECIAL RELATIVITY to work, we need inertial frames. Can we find inertial frames in the presence of gravity?

Thought experiment:

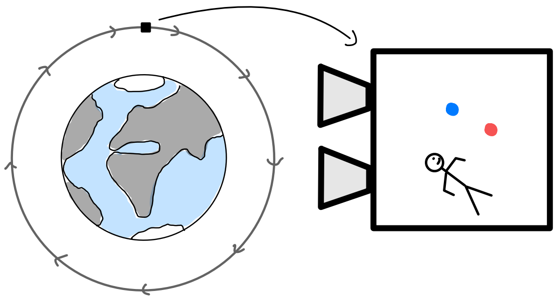
a | ⊲ Laboratory on the surface of Earth:



→ Not an inertial system ☹

The problem is that we cannot simply shroud our lab by some magic material that shields the gravitational force. By contrast, this *can* be done for the electromagnetic field (↓ *Faraday cage*, ↑ *Mu-metal*). Note that this is not a technical problem, it is a fundamental one!

b | ⊲ Interior of orbital space station:

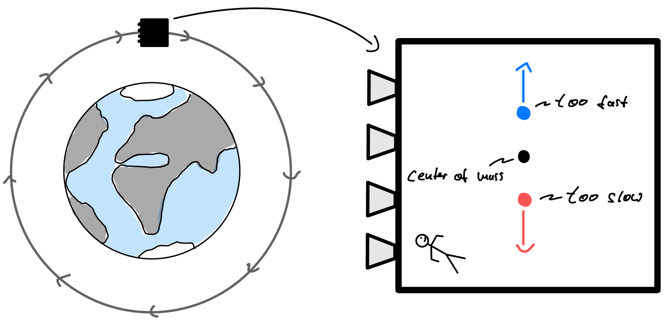


→ Approximate inertial system ☺

The space station is equivalent to a free falling laboratory, where the gravitational force is canceled exactly by inertia. What makes a space station so convenient is that it also has orbital velocity so that it "falls around Earth" and therefore can be used much longer than a free falling lab that eventually crashes on the surface.

This is a situation where it actually makes sense to use the terms "gravity" and "gravitation" differently: In the space station, there is no *gravity* (the astronauts float), but there is a *gravitational field*! (The latter is just canceled by the inertial force due to free fall.) This situation is different from being in a spaceship far away from Earth in interstellar space where no gravitational forces can be measured. (Although these two situations cannot be distinguished from within a small space station/spaceship, → *later*.)

c | ⊲ Very large orbital space station:

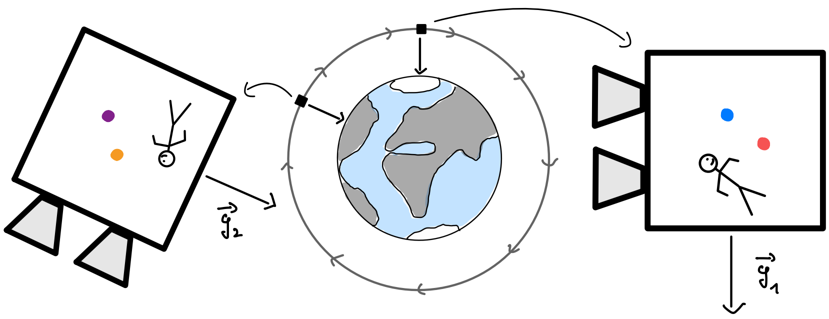


↑ *Tidal forces* → Not an inertial system ☹

When we extend the size of the space station, the inhomogeneity of the gravitational acceleration becomes noticeable and the "inertial test" `IN` fails. Inhomogeneous gravitational fields therefore constrain the size (both in space and time) of reference systems that satisfy the properties of an inertial system. Hence our assumption that inertial systems cover all of spacetime (and therefore can describe arbitrary physical phenomena, not just local ones) is invalidated by the presence of inhomogeneous gravitational fields.

Note that there is another way to detect the inhomogeneous gravitational field and make the system non-inertial: Stay in the small spacecraft and wait longer. At some point you will notice that the two balls drift apart—even when they are only centimeters apart. This shows that the approximate inertial system really must be small in space *and time*.

d | ⊲ Two small orbital space stations:



→ Local inertial systems accelerated wrt. each other ☹

If you imagine that these small inertial systems overlap on their boundaries, you could ask how to transform the coordinates of an event in this overlap from one of these systems into the other. Because these systems are accelerated wrt. each other, this transformation cannot be linear, in particular it cannot be a Lorentz transformation! There seems to be something missing; what determines this transformation?

iii | This thought experiment leads us to the following (troubling) conclusion:

> Inertial systems can only exist *locally* in an inhomogeneous gravitational field. How to transform between these local inertial systems is unclear (→ *later*).

→ Extended phenomena cannot be described by SPECIAL RELATIVITY!

"Extended" here means "on the scale of gravitational inhomogeneities."

11 | In a nutshell:

> SPECIAL RELATIVITY cannot …
>
> - … describe the gravitational field itself.
> - … describe physics in inhomogeneous gravitational fields.
>
> How to fix this? → GENERAL RELATIVITY

8.3. ‡ The gravitational redshift and curved spacetime

Quite surprisingly, one can derive some predictions of GENERAL RELATIVITY without knowledge of the detailed theory. One is the → *gravitational redshift* of light, which, again without the usage of heavy math, implies that GENERAL RELATIVITY must describe a *curved* spacetime.

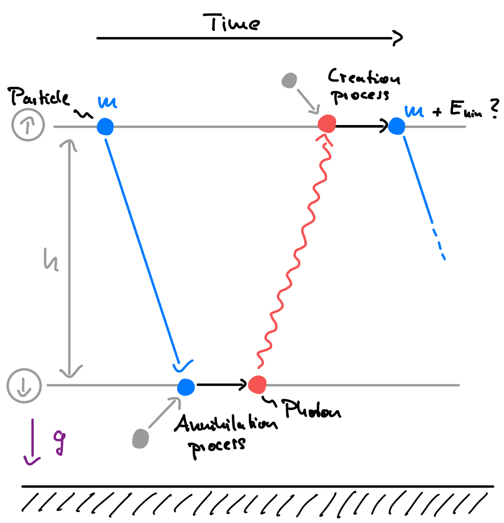
The following is based on Sections 7.2 and 7.3 of MISNER *et al.* [2] and Section 2.1 of CARROLL [102].

1 | Gravitational redshift:

EINSTEIN already concluded in 1908 that light leaving a gravitational potential must be redshifted [95]. The following he showed in 1911 [103], that is, years before he finalized GENERAL RELATIVITY.

i | ∢ Particle of rest mass m in (homogeneous) gravitational potential:

In the following, we assume that *inertial* and *gravitational* mass are equal: $m_\mathrm{I} = m = m_\mathrm{G}$.



ii | Step 1: Drop the particle by h → New total energy:

$$E_{\text{tot}} = m_{\text{I}}c^2 + m_{\text{G}}gh = m(c^2 + gh) \tag{8.11}$$

iii | Step 2: Assume electron annihilates into a photon with energy:

$$E_\downarrow = E_{\text{tot}} \tag{8.12}$$

iv | Step 3: Let the photon propagate upwards by h → New photon energy $E_\uparrow$:

- Possibility 1: Photon is *not* affected by gravity → $E_\uparrow = E_\downarrow > mc^2$ ✗

  This immediately leads to a violation of energy conservation because the photon can now be used to recreate the particle *plus some kinetic energy* that wasn't there before.

- Possibility 2: Photon is *redshifted* by gravitational field such that $E_\uparrow = mc^2$ ✓

  This is the only possibility consistent with energy conservation, i.e., the photon must loose energy just as a particle would when climbing the potential.

v | Thus we find for the photon energies:

$$E_\downarrow = \frac{E_\uparrow}{c^2}(c^2 + gh) = E_\uparrow\left(1 + \frac{gh}{c^2}\right) \tag{8.13}$$

vi | ⊲ ⁂ *Redshift parameter* $z := \Delta\lambda/\lambda = (\lambda_\uparrow - \lambda_\downarrow)/\lambda_\downarrow$

The redshift z measures the relative change in wavelength $\Delta\lambda$ wrt. a reference wavelength λ.

→ Gravitational redshift: (Use the photon energy $E = h\nu = hc/\lambda$.)

$$\boxed{1 + z = \frac{\lambda_\uparrow}{\lambda_\downarrow} = \frac{E_\downarrow}{E_\uparrow} = 1 + \frac{gh}{c^2}} \tag{8.14}$$

Using the ↑ *Mößbauer effect*, Robert Pound and Glen Rebka verified this prediction in 1960 with their famous ↑ *Pound-Rebka experiment* [104, 105].

2 | Schild's argument:

The following reasoning goes back to Alfred Schild (see references in [2]) and demonstrates that a relativistic theory of gravity cannot be formulated on a flat Minkowski spacetime:

i | Assumptions:

- There exists an extended inertial frame K attached to Earth's center.
  (We relax our definition and allow particles to be accelerated near earth.)
- In this frame, proper time and lengths are given by the Minkowski metric.
- There is some gravitational field (of unspecified nature) that matches observations.
  (This implies the gravitational redshift derived above.)

ii | Thought experiment:

a | ⊲ Two observers $O_{\downarrow,\uparrow}$ at height $z_{\downarrow,\uparrow}$ with $z_\uparrow = z_\downarrow + h$ at rest in K

b | Observer $O_\downarrow$ emits a light signal with wavelength $\lambda_\downarrow$

→ Time for one wavelength: $\Delta t_\downarrow = \lambda_\downarrow/c$

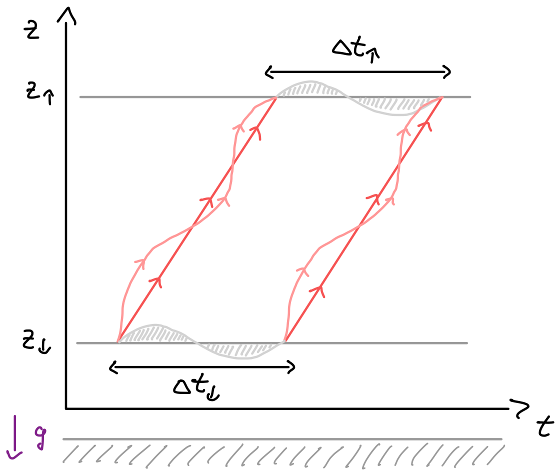
Note that since both observers are at rest in K, their proper times and the coordinate time of K coincide.

c | Observer $O_\uparrow$ receives the signal with wavelength $\lambda_\uparrow$

→ Time for one wavelength: $\Delta t_\uparrow = \lambda_\uparrow / c$

d | Redshift $\rightarrow \lambda_\uparrow > \lambda_\downarrow \rightarrow \Delta t_\uparrow > \Delta t_\downarrow$

e | But in the Minkowski diagram of the (imagined) global inertial frame, the experiment looks as follows:

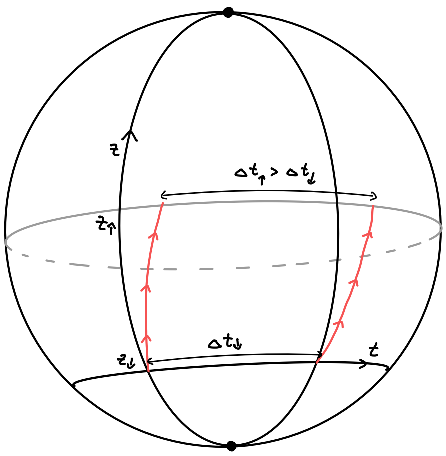


$\rightarrow \Delta t_\uparrow = \Delta t_\downarrow$ ↯

Note that the only important aspect for the contradiction is that the two world lines of the start and end of one wavelength are *congruent* in Minkowski space. That is, we do not need to know how gravity affects the trajectory of light (maybe it is bent). The only important thing is that both trajectories are bent in the same way, which is to be expected in a static scenario where the gravitational field does not change.

iii | Conclusion:

In the presence of gravity, the trajectories of light signals in spacetime must be congruent (if they are straight: parallel)—but at the same time their distance in time direction must change! This is impossible in the flat (pseudo-)Euclidean geometry of Minkowski space; but it *is* possible in a curved spacetime. As we will see → *later*, the tendency of initially parallel "straight lines" (→ *geodesics*) to approach or recede from another is exactly what characterizes a curved space(time):



→ Spacetime must be curved!

3 | We can summarize:

A Lorentz covariant theory of gravity cannot be formulated on Minkowski space.

This already suggests that we will need the more general machinery of differential geometry, introduced in Chapter 3, to model spacetime not as flat Minkowski space, but as a more general, curved pseudo-Riemannian manifold.

9. Conceptual Foundations

9.1. Einstein's equivalence principle

The wording of the equivalence priciples are paraphrased from CARROLL [102].

1 | Remember:

There are two concepts of mass in Newtonian physics:

$$\text{Inertial mass } m_{\mathrm{I}}: \quad \vec{F} = m_{\mathrm{I}}\vec{a} \tag{9.1a}$$

$$\text{Gravitational mass } m_{\mathrm{G}}: \quad \vec{F} = -m_{\mathrm{G}}\nabla\phi \tag{9.1b}$$

Strictly speaking, two gravitational masses must be conceptually distinguished: The *passive* gravitational mass is the charge that couples to the gravitational field via Eq. (9.1b). The *active* gravitational mass is the source of the gravitational potential $\phi = -GM_{\mathrm{G}}/r$. However, given that Newton's third law is valid (action equals reaction), the situation is completely symmetric and these two masses can be identified. Thus, in the following we only distinguish between inertial and gravitational mass.

→ Gravitational acceleration:

$$a_{\mathrm{G}} = \frac{m_{\mathrm{G}}}{m_{\mathrm{I}}} \underbrace{\frac{GM_{\oplus}}{r^2}}_{g} \tag{9.2}$$

It has been long known (since Galileo Galilei) that the gravitational acceleration is *independent* of the material of the body (if one can ignore air resistance): *All bodies fall at the same rate.*

Experience:

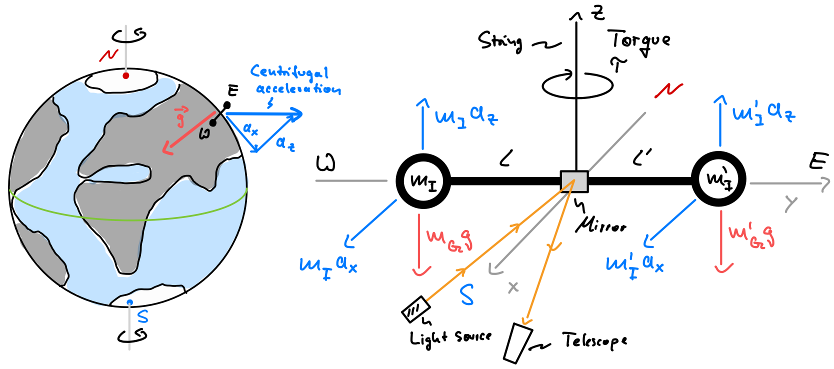
$$\frac{m_{\mathrm{G}}}{m_{\mathrm{I}}} = \text{const} \quad \rightarrow \quad \text{Choose units appropriately:} \quad \frac{m_{\mathrm{G}}}{m_{\mathrm{I}}} = 1 \tag{9.3}$$

In classical mechanics, this is just an observation; it is neither explained nor necessary for its consistency.

2 | The Eötvös experiment [106]: (See also the later publication [107].)

While there have been earlier experiments that quantitatively tested the equivalence of inertial and gravitational mass, the experiment by Hungarian physicist ROLAND EÖTVÖS made huge improvements in precision. The experiment was used by Einstein as an argument for his → *equivalence principle*.

⊲ Torsion balance with two different test bodies: (Details: ➔ Problemset 1)



➔ Problemset 1 $\overset{\circ}{\rightarrow}$

$$\text{Torque:} \quad \tau \approx \underbrace{m_G a_x l}_{\neq 0} \left(\frac{m_I}{m_G} - \frac{m_I'}{m_G'} \right) \tag{9.4a}$$

$$\tau = 0 \quad \Leftrightarrow \quad \underbrace{\frac{m_I}{m_G}}_{\text{Material A}} = \underbrace{\frac{m_I'}{m_G'}}_{\text{Material B}} = \text{const} \tag{9.4b}$$

→ Result by Eötvös [106]:

$$\frac{\delta m}{m} = \frac{m_I - m_G}{m_I} < 3 \times 10^{-9} \tag{9.5}$$

The latest (2022) and most precise results testing the equivalence of inertial and gravitational mass come frome the satellite-based MICROSCOPE experiment [108]; it improved the upper bound for a violation of the equivalence to $\delta m/m < 10^{-15}$. Recent experiments also demonstrated the equivalence for antimatter [109].

→ Experimental fact:

$$\boxed{\text{Inertial mass and gravitational mass are proportional } (\textit{w.l.o.g. } m_I = m_G).} \tag{9.6}$$

This trivial sounding assertion (when have you ever distinguished between these two masses?) has profound consequences: Recall that SPECIAL RELATIVITY is concerned with the concept of *inertia* (e.g. by using inertial systems); in particular, the (rest) mass that shows up in the mass-energy equivalence $E_0 = mc^2$ is the inertial mass m_I of the system. The equivalence above now links this mass to the gravitational mass, and therefore asserts the the inert bodies of SPECIAL RELATIVITY must be affected by (and be sources of) gravity. But SPECIAL RELATIVITY had nothing to say about gravity! Quite to the contrary: as discussed in Section 8.2, the theory cannot accomodate gravity in a consistent way.

3 | Classical mechanics does not explain Eq. (9.6). However, if we take Eq. (9.6) for granted, a homogeneous gravitational field vanishes in an accelerated frame:

⊲ N particles of (inertial = gravitational) mass m_k in gravitational field:

$$\underbrace{m_k \frac{\mathrm{d}^2 \vec{x}_k}{\mathrm{d}t^2}}_{\text{Inertia}} = \underbrace{m_k \vec{g}}_{\text{Gravity}} + \sum_{l \neq k} \vec{F}_{kl}(\vec{x}_k - \vec{x}_l) \quad \text{with } k = 1, \dots, N. \tag{9.7}$$

⊲ Coordinate transformation into free-falling frame:

$$t' = t \quad \text{and} \quad \vec{x}'_k = \vec{x}_i - \tfrac{1}{2}\vec{g}t^2 \,. \tag{9.8}$$

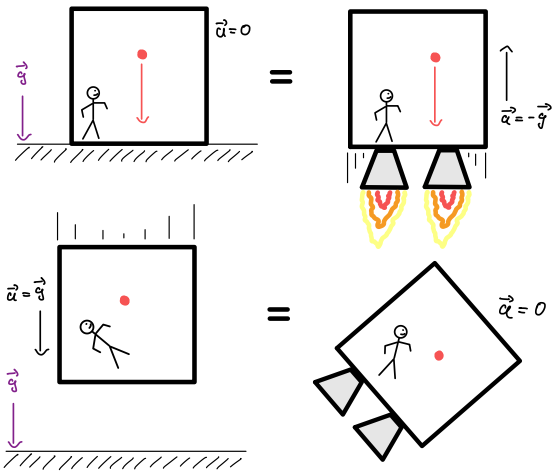
This coordinate transformation is *non-linear*, in particular, it is not a Galilei transformation!

$\overset{\circ}{\rightarrow}$ Equations of motion in the free-falling coordinate system:

$$m_k \frac{\mathrm{d}^2\vec{x}'_k}{\mathrm{d}t'^2} = \sum_{l\neq k} \vec{F}_{kl}(\vec{x}'_k - \vec{x}'_l) \quad \text{with } k = 1, \ldots, N\,. \tag{9.9}$$

→ No graviy in the free-falling frame!

This matches our experience and can be illustrated with the following though experiment:

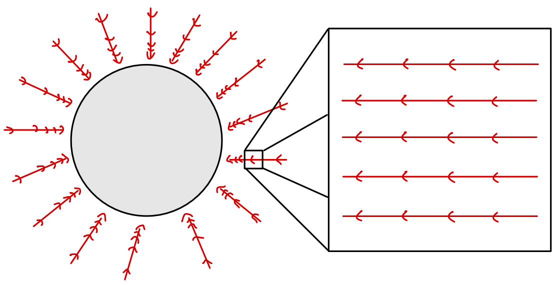


4 | Caveat: Only true for *homogeneous* gravitational fields: $m_k\vec{g}$.

What about *inhomogeneous* gravitational fields?

Note that the inhomogeneity of gravity is essential for planets and stars to form; it is the root cause for complexity in the world that is neccessary for life to exist. This is not a slight inconvenience we can sweep under the rug!

→ Small enough regions look homogeneous:



→ Gravity can be compensated *locally* in an accelerated frame.

¡! This implies that there is no transformation to a *global* free-falling reference frame in which inhomogeneous gravitational fields vanish. Thus acceleration and gravity are only equivalent *locally*; globally, they are physically distinct. In particular, this means that the phenomenon of gravity is not just "acceleration in disguise." As mentioned previously, accelerated coordinate systems (and bodies) are something that SPECIAL RELATIVITY can handle. If gravity and acceleration were equivalent globally, SPECIAL RELATIVITY would be sufficient to describe gravity and there was no need for GENERAL RELATIVITY.

5 | Given all these facts, it is reasonable to proclaim the following principle:

§ Postulate 1: Weak equivalence principle WEP **(*Univesality of free fall*)**

In small enough regions of spacetime, the motion of *freely-falling particles in a gravitational field* and *free particles in a uniformly accelerated frame* are the same.

(This formulation formalizes the idea sketched in the upper panel of the sketch above.)

Equivalently:

For every event, there is a local reference frame, covering small enough regions of spacetime in its vicinity, such that gravity has no effect on the motion of arbitrary particles in this frame and the law of inertia holds.

(This formulation formalizes the idea sketched in the lower panel of the sketch above.)

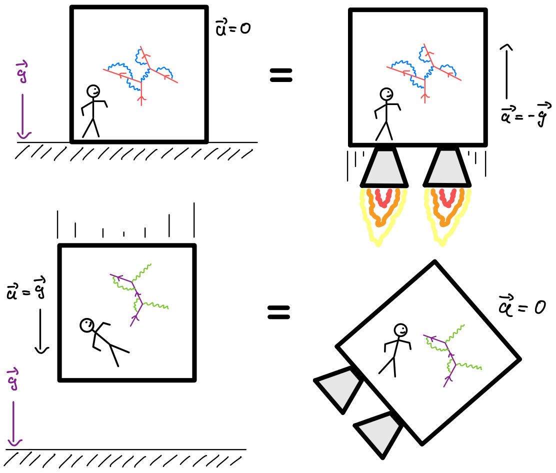
6 | Einstein's generalization:

The local equivalence of gravity and accelerated frames is true for *all* physical phenomena (and not only classical mechanics).

Einstein was aware of the Eötvös experiment and was convinced that the equivalence of inertial and gravitational mass hinted at a deep relationship between inertia (acceleration) and gravitation. He wrote in 1907 [95] (highlights are mine):

> *Bisher haben wir das Prinzip der Relativität, [..], nur auf beschleunigungsfreie Bezugssysteme angewendet. Ist es denkbar, daß das Prinzip der Relativität auch für Systeme gilt, welche relativ zueinander beschleunigt sind? [..]*
>
> *Wir betrachten zwei Bewegungssysteme Σ_1 und Σ_2. Σ_1 sei in Richtung seiner X-Achse beschleunigt, und es sei γ die (zeitlich konstante) Größe dieser Beschleunigung. Σ_2 sei ruhend; es befinde sich aber in einem homogenen Gravitationsfelde, das allen Gegenständen die Beschleunigung $-\gamma$ in Richtung der X-Achse erteilt.*
>
> *Soweit wir wissen, unterscheiden sich die physikalischen Gesetze in bezug auf Σ_1 nicht von denjenigen in bezug auf Σ_2;* es liegt dies daran, daß alle Körper im Gravitationsfelde gleich beschleunigt werden. *Wir haben daher bei dem gegenwärtigen Stande unserer Erfahrung keinen Anlaß zu der Annahme, daß sich die Systeme Σ_1 und Σ_2 in irgendeiner Beziehung voneinander unterscheiden,* und wollen daher im folgenden die völlige physikalische Gleichwertigkeit von Gravitationsfeld und entsprechender Beschleunigung des Bezugssystems annehmen.

Pictorially, Einstein claims that *any type* of local experiment cannot distinguish gravity from acceleration (here, for example, some quantum mechanical scattering process):



7 | We can formalize this as follows:

§ Postulate 2: Einstein's equivalence principle EEP

In small enough regions of spacetime, the laws of physics reduce to those of SPECIAL RELATIVITY: It is impossible to detect the existence of a gravitational field by means of local (non-gravitational) experiments.

Equivalently:

For every event, there is a local reference frame, covering small enough regions of spacetime in its vicinity, such that gravity has no effect *on any (non-gravitational) experiment* in this frame and the law of inertia holds.

- The EEP implies the WEP.
- Excluding non-gravitational experiments means that the intrinsic gravitational energy of our experiment does not contribute significantly to its mass (see SEP below). Note that we do not require that gravitational experiments (using large masses) *can* locally distinguish between gravity and acceleration; the WEP does simply not constraint such experiments.
- It is important to appreciate the profound implications of this principle for doing physics in a gravitational field: It asserts that as long as your laboratory is small (compared to the inhomogeneties of the gravitational field) and free-falling (e.g. a space station in orbit), SPECIAL RELATIVITY is sufficient to describe *all experiments* that you can conduct in this lab. In particular, the fact that SPECIAL RELATIVITY cannot describe gravity is not important because in the free-falling lab there is none. This means that everything we discussed last term remains valid—and therefore useful—*locally*. Thus gravity does not completely invalidate SPECIAL RELATIVITY, it only restricts its domain of validity to local, free-falling inertial frames. I hope you are happy to hear that!
- More precisely, for every event (point in spacetime) there is an *equivalence class* of local inertial frames (related by boosts), equiped with inertial coordinate systems (related by translations and rotations), in all of which SPECIAL RELATIVITY holds good. The coordinate transformations between these systems are given by Lorentz transformations. (You can check

the existence of such frames in our Newtonian calculation above by adding a term $\vec{v}t$ to the transformation of the position coordinates.)

- In our mathematical framework of differential geometry, the equivalence class of local inertial systems at a spacetime point will be identified with the ← *tangent space* of the spacetime manifold at that point.

8 | Concerning *gravitational* laws of physics:

In our defintion of the EEP, we excluded experiments that depend on the gravitational interaction itself (e.g., use objects with considerable intrinsic gravitational energy). This exclusion follows SCHRÖDER [3], whereas other authors like CARROLL [102] include the (unkown) gravitational laws of physics in the EEP.

For us, it makes then sense to define an extension of the EEP as follows:

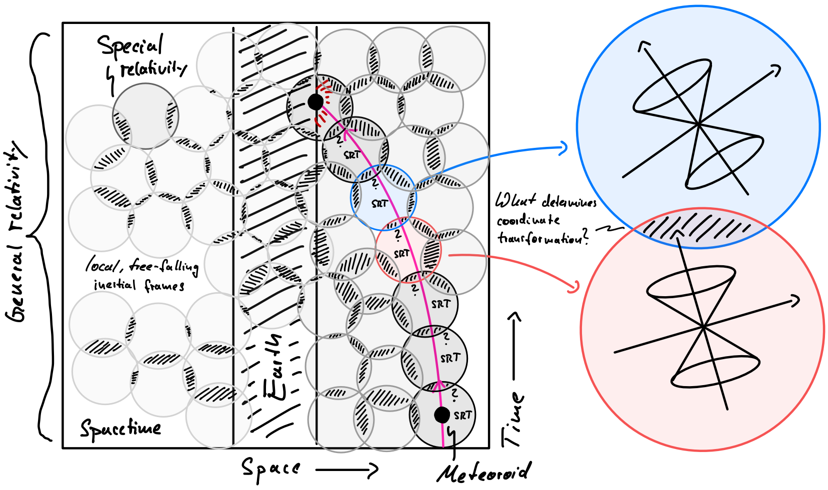
> **§ Postulate 3: Strong equivalence principle** SEP
>
> The EEP is valid for all laws of physics, including the gravitational laws.

- GENERAL RELATIVITY satisfies the SEP (and thereby the EEP and the WEP).
- The reason to separate the EEP from the SEP is that alternatives to GENERAL RELATIVITY can *satisfy* the EEP (and the WEP) but *violate* the SEP. This alternatives can be metric theories like GENERAL RELATIVITY with additional fields; see Ref. [110] for details.
- In particular, the SEP requires that the universality of free fall (WEP) also holds for large bodies like planets (not just small test particles) with significant amounts of gravitational self-energy. More precisely, the SEP demands that the rest mass E_{grav}/c^2 that comes from the gravitational self-energy E_{grav} accelerates just like any other rest mass in an external gravitational field.

  Note that the validity of the SEP cannot be deduced from typical experiments that test the WEP because these experiments use small test masses with a gravitational self-energy that is way too small to detect any violation of the SEP (because gravity is such a weak force). One needs to use planet-sized objects to draw conclusions about the SEP (→ *next*).
- Using reflectors on the moon (left by Apollo 11 in 1969), lunar laser ranging (LLR) can be used to experimentally test the SEP since the fractions of gravitational self-energy of moon and earth are different enough to modify moon's orbit measurably if the SEP was violated. To date there is no evidence of such a violation to high precision [111, 112], hence we will assume that the SEP holds.

9 | If SPECIAL RELATIVITY explains *everything* you can do in a local, free-falling laboratory, at which point does gravity enter the picture? Well, the hitch is that not all physical processes *can* be restricted to a single, local inertial frame:

⊲ Meteroid traversing the inhomogeneous gravitational field of earth:



How to model the trajectory?

Imagine you start in a local inertial system where you know the initial data (position, velocity) of the meteroid. Since SPECIAL RELATIVITY is valid in this small patch, you can use the known equations of relativistic mechanics to compute the trajectory of the meteroid. However, at some point, the meteroid will leave the local inertial system and enter another one. To proceed with your application of relativistic mechanics, you need to know the coordinate transformation that maps the coordinates of the final position and velocity in the first inertial system to the coordinates of the next.

But a priori these inertial system are unrelated, in particular, they can be accelerated with respect to one another (recall the two small space stations in Section 8.2). To proceed with your application of relativistic mechanics, you need this coordinate transformation! And this is where gravity hides: The gravitational field (here generated by earth) and the pattern of local coordinate transformations are one and the same thing! This is what is meant by *gravity becoming a geometric property of spacetime*.

→

> The gravitational field is the (dynamical) structure that determines which local frames of reference are inertial or, equivalently, how to transform from one local inertial frame to the next.