↓Lecture 16 [09.04.24]

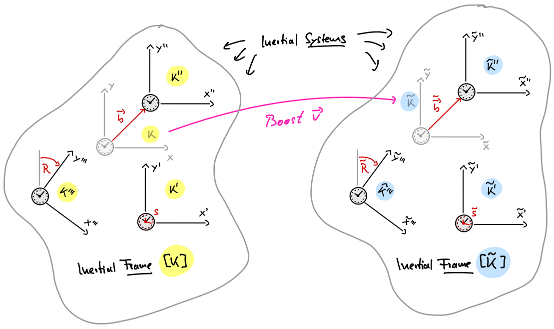
8. Limitations of SPECIAL RELATIVITY

8.1. Reminder: SPECIAL RELATIVITY

1 | SPECIAL RELATIVITY in a nutshell:

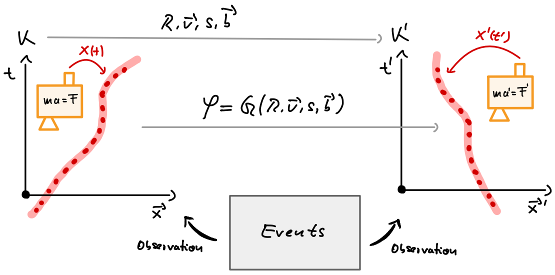
- ← *Inertial frames* [Section 1.1]

  There exists a special class of infinitely extended reference frames (equipped with Cartesian coordinates) in which the law of inertia holds (IN = the trajectories of free particles are straight lines that are traversed with constant velocity). All inertial frames move relative to each other with constant velocities:



- ← *Einstein's principle of (special) relativity* SR [Section 1.3]

  The laws of physics (orange boxes in the sketch below) have the same form in all inertial systems. This extends Galilei's principle of relativity which makes this claim only for the realm of mechanics. The modifier "special" emphasizes that the principle makes only claims about the special class of inertial systems:



We characterized SR previously as follows: *No experiment can distinguish between inertial frames.* This description can be misleading, so let me prevent any misconceptions: What we mean is that there is no local physical experiment that you can perform in a sealed box (at rest in some inertial system) that allows you to figure out in which inertial system your box is at rest. In this way you probe the *form of physical laws* (e.g., whether there is an additional Coriolis term in your equation of motion or not) and thus probe the validity of SR as formulated above.

The statement above does *not* mean that there is no operational way to label specific inertial systems. For example, we can define the (approximate) inertial frame in which the center of Earth is at rest and, for comparison, another inertial frame in which the ↑ *cosmic microwave background (CMB)* has no dipole structure (the latter has a velocity of roughly 360 km s^{-1} wrt. the former). Clearly there are experiments to decide whether you are in one or the other (measure the CMB dipole and/or the velocity of Earth relative to you). This does *not* violate SR though, because all phenomena you observe in these frames of reference are described correctly *by the same equations* (e.g. you can use the same Maxwell equations to describe the CMB radiation in both inertial frames). This is also why the existence of the global inertial frame labeled by a CMB without dipole is not in conflict with SPECIAL RELATIVITY. There is a difference between *physical states* and *physical laws*; SR only makes claims about the latter.

- ← *Lorentz transformations* [Section 1.5]

  The coordinate transformations that map the record of physical events from one inertial system to another are given by Lorentz transformations (more generally: Poincaré transformations). (Proper orthochronous) Lorentz transformations form a group $\mathrm{SO}^+(1,3)$ and are parametrized by a three-dimensional rotation and a three-dimensional boost velocity. They linearly map the spacetime coordinates $(t,\vec{x})$ of an event in one inertial system K to the spacetime coordinates $(t',\vec{x}')$ of the same event in another inertial system K'.

  A pure boost in x-direction has the form:

$$\Lambda(K \xrightarrow{v_x} K') : \begin{cases} ct' = \gamma\left(ct - \frac{v_x}{c}x\right) \\ x' = \gamma(x - v_x t) \\ y' = y \\ z' = z \end{cases} \quad \text{with Lorentz factor} \quad \gamma = \frac{1}{\sqrt{1 - v_x^2/c^2}} \tag{8.1}$$

- ← *Constancy of the speed of light* SL [Section 1.5]

  Lorentz transformations are characterized (and differ from Galilei transformations) by the existence of a *finite maximum velocity* $v_{\max}$. Experience tells us to identify this velocity with the speed of light c. Lorentz transformations then imply that this maximum velocity is the same for all initial observers (← *relativistic addition of velocities*):

$$\text{Experiments} \rightarrow \quad v_{\max} < \infty \quad \Leftrightarrow \quad \text{Lorentz transformations} \tag{8.2a}$$

$$v_{\max} = \infty \quad \Leftrightarrow \quad \text{Galilei transformations} \tag{8.2b}$$

- ← *Tensor calculus* [Chapter 3 and Chapter 4]

  Combining the principle of relativity with the assertion that Lorentz transformations translate between inertial systems implies that the laws of nature must be expressed as equations that are forminvariant under Lorentz transformations (← *Lorentz covariance*). The Lorentz covariance of a theory can be quite tedious to show and even more tedious to ensure when constructing it from scratch. (Recall Maxwell equations in their conventional form!) This is why we prefer equations in which the Lorentz covariance is *manifest*. To achieve this, we developed tensor calculus as a "toolbox" to construct Lorentz covariant equations from Lorentz scalars, vectors, tensors, ….

  For example, the equation of motion for a charged particle in an electromagnetic field reads and transforms as follows:

$$\underbrace{\frac{\mathrm{d}p_i^\mu}{\mathrm{d}\tau} = \frac{q_i}{c} F^\mu{}_\nu\, u_i^\nu}_{\text{Inertial system } K} \xrightarrow[]{\substack{\bar{p}^\mu = \Lambda^\mu{}_\nu p^\nu \\ \bar{u}^\mu = \Lambda^\mu{}_\nu u^\nu \\ \bar{F}^\mu{}_\nu = \Lambda^\mu{}_\alpha \Lambda_\nu{}^\beta \bar{F}^\alpha{}_\beta}} \underbrace{\frac{\mathrm{d}\bar{p}_i^\mu}{\mathrm{d}\tau} = \frac{q_i}{c} \bar{F}^\mu{}_\nu\, \bar{u}_i^\nu}_{\text{Inertial system } \bar{K}} \tag{8.3}$$

2 | Problems:

Despite the undeniable success of SPECIAL RELATIVITY, it's not just sunshine and roses:

- What about gravitation?

  In our discussion of SPECIAL RELATIVITY we explicitly avoided the phenomenon of gravitation (we will see below why). This makes SPECIAL RELATIVITY clearly incomplete (and *special*) as a description of nature (which, on very large scales, is dominated by the gravitational force) and asks for a more *general* theory.

- Why are inertial coordinate systems special?

  SPECIAL RELATIVITY describes physics with respect to a particular class of reference frames (inertial frames) in a particular class of coordinates (Cartesian coordinates). Only in these coordinate frames the laws of nature take their "simplest" form and the Lorentz transformation only translates between these special coordinate systems. However, in our very general discussion of differential geometry (Chapter 3) we established the notion of "geometric objects" that are independent of coordinates. We also interpreted coordinates as mathematical auxiliary structures to label events, and denied their physical existence ("coordinates do not exist"). SPECIAL RELATIVITY does not live up to this rather fundamental claim with its focus on inertial coordinate systems. Shouldn't there be a formulation of physics in which coordinates play no role at all?

- What is the origin of inertia?

  Remember Newton's bucket (p. 13)? It's punchline was to argue for the existence of an entity ("absolute space") which determines whether an object is accelerated or not. SPECIAL RELATIVITY, of course, disposes of Newton's absolute space wrt. to which position and velocity can be measured (no *ether*!). The existence of such, however, was never implied by the bucket experiment anyway, which asks about the absolute notion of *acceleration*. And SPECIAL RELATIVITY is silent about the origin of inertia and what determines whether the water in Newton's bucket is concave or flat (we simply assumed that inertial frames exist, we neither asked where they come from nor what makes them inertial in the first place). This situation is clearly unsatisfactory.

3 | Non-problems:

Sometimes one hears that *acceleration* is a problem for SPECIAL RELATIVITY. This is not so:

- Accelerated motion ✓

  SPECIAL RELATIVITY of course describes accelerated objects perfectly well. Recall our concept of 4-acceleration in Section 5.1, the relativistic equation of motion in Eq. (5.6), and the validity of the proper time integral for arbitrary time-like trajectories in Eq. (2.25). Note that these equations are only valid in inertial frames, though.

- Accelerated observers ✓

  While our equations were given in inertial systems (where, according to Einstein's principle of relativity, the laws of physics take the same *and simplest* form), SPECIAL RELATIVITY *can* describe the physics in accelerated non-inertial frames as well (e.g. using the concept of instantaneous rest frames). In such non-inertial coordinate systems the physical laws do *not* take their simplest forms and can look messy (in particular, one cannot Lorentz transform into these frames). This, however, does not mean that we cannot describe what happens in such systems. As an example recall the relativistic rocket of ➔ Problemset 6. (For details see Chapter 6 in MISNER *et al.* [2] and also Einstein's original work [95].)

8.2. The special role of gravity

Let us now focus on the problem of incorporating gravity into SPECIAL RELATIVITY. It is important to understand why the gravitational force poses a fundamental problem for the framework of SPECIAL RELATIVITY (and is not just a technical inconvenience).

Note on nomenclature:

In English, there are two terms with slightly different meaning (if we take *Merriam-Webster* as a reference):

> ***Gravity:*** *the gravitational attraction of the mass of the Earth, the moon, or a planet for bodies at or near its surface*
>
> ***Gravitation:*** *a force manifested by acceleration toward each other of two free material particles or bodies or of radiant-energy quanta*

This distinction has no counterpart in German as far as I can tell (perhaps "Schwerkraft" vs. "Gravitation"?). Given that even the English literature does not seem to be consistent, I will use these two terms interchangeably. Their context will suffice to establish semantic clarity.

4 | Recall Newton's law of universal gravitation:

i | $\sphericalangle$ Mass distribution $\rho(\vec{x}) \rightarrow$

$$\nabla^2\phi(\vec{x}) = 4\pi G\rho(\vec{x}) \quad \rightarrow \quad \text{Gravitational potential } \phi(\vec{x}) \tag{8.4}$$

$G \approx 6.674\,301 \times 10^{-11}\,\mathrm{m^3 s^{-2}/kg}$: Gravitational constant

Please appreciate the smallness of G (and therefore the weakness of gravity) as compared to the human-scale units in which it is given. Gravity is, if compared to the other three fundamental forces, by far (really really really far) the weakest force. It is a fundamental unsolved problem of physics why this is so.

$\rightarrow$ Equation of motion of test mass (e.g. a satellite):

$$m_\mathrm{I}\ddot{\vec{r}} = -m_\mathrm{G}\nabla\phi(\vec{r}) \tag{8.5}$$

m_I: inertial mass
m_G: gravitational mass

We will discuss the relation of m_I and m_G later (Section 9.1).

ii | Example: Static point mass $M_\mathrm{G} \gg m_\mathrm{G}$ as source in origin (e.g. Earth):

$$\phi(\vec{x}) = -\frac{GM_\mathrm{G}}{|\vec{x}|} \quad \Rightarrow \quad m_\mathrm{I}\ddot{\vec{r}} = \vec{F}_\mathrm{G} = -G\frac{m_\mathrm{G}M_\mathrm{G}}{r^2}\hat{r} \tag{8.6}$$

If the source is dynamic as well, $\vec{r}$ is the relative distance vector between the two masses and m_I must be replaced by the reduced mass of the two bodies.

iii | Observation: Equations [especially Eq. (8.4)] are not Lorentz covariant!

You can check that they are ← *Galilei invariant*, recall Eq. (1.18).

This is no surprise: We already know from our discussions in Section 6.4 that in RELATIVITY, classical forces can only act *locally*, and not at a distance. Interactions between distant objects must be mediated by dynamical degrees of freedom (a "field") to obey the speed limit for information propagation imposed by Lorentz symmetry. But Newton's gravitational potential ϕ is static and not dynamic!

5 | Problem: "Action at a distance" (Gravitational force acts instantaneously and has no dynamics.)

Isaac Newton writes in a letter to Bentley in 1692 [96]:

> *It is inconceivable, that inanimate brute Matter should, without the Mediation of something else, which is not material, operate upon, and affect other Matter without mutual Contact, as it must be, if Gravitation in the Sense of Epicurus, be essential and inherent in it. And this is one Reason why I desired you should not ascribe innate Gravity to me. That Gravity should be innate, inherent and essential to Matter, so that one Body may act upon another at a Distance through a Vacuum, without the Mediation of any thing else, by and through which their Action and Force may be conveyed from one to another, is to me so great an Absurdity, that I believe no Man who has in philosophical Matters a competent Faculty of thinking, can ever fall into it. Gravity must be caused by an Agent acting constantly according to certain Laws; but whether this Agent be material or immaterial, I have left to the Confederation of my Readers.*

Thus even Newton himself was not entirely satisfied with his law of universal gravitation (which describes an action at a distance) and anticipated some entity that mediates the force.

6 | First try: Make gravitational potential a *dynamic* field:

$$\text{Poisson equation Eq. (8.4)} \quad \rightarrow$$

$$\text{Wave equation:} \quad \partial^2\phi(t,\vec{x}) = \left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right)\phi = -4\pi G\,\rho(t,\vec{x}) \tag{8.7}$$

→ Gravity propagates with the speed of light ☺

$\overset{*}{\rightarrow}$ Problems:

For a detailed study of a fully specified scalar theory of gravity: ➔ Problemset 1 (also Exercise 7.1 in MISNER *et al.* [2]). See also Ref. [97].

- Electromagnetic field cannot couple to gravity → No bending of light ☹

  Today it is a well tested fact that light follows a curved trajectory in strong gravitational fields (→ *later*). Thus any theory that does not couple the EM field to gravity must be incorrect.

  Here is a quick-and-dirty explanation why a theory of the form Eq. (8.7) fails to couple to the electromagnetic field in a relativistic setting:

  Since ϕ is assumed to be a scalar field, for Eq. (8.7) to be Lorentz covariant, ρ must be a scalar as well. In a relativistic theory, energy and (inertial) mass are equivalent ($E_0 = mc^2$). If we assume that gravitational mass and inertial mass are equivalent (→ *later*), this implies that energy (density) must be a source of gravity. The problem is that the energy (density) (of any theory) is the 00-component of the ← *energy-momentum tensor* T^{00} (this is the charge density associated to the Noether current that comes from translation symmetry in time); in particular, the energy density is *not* a scalar and therefore cannot be used as a source on the right-hand side of Eq. (8.7). The only scalar we can construct from the energy-momentum tensor is the ⁂ *Laue scalar* $T = T^\mu{}_\mu = \eta_{\mu\nu}T^{\mu\nu}$, i.e., the trace of the EMT. Thus a simplistic but fully Lorentz covariant form of scalar gravity is

$$\partial^2\phi = -\tfrac{4\pi G}{c^2}\,T\,. \tag{8.8}$$

(For a complete theory one also needs a Lorentz covariant analog of Eq. (8.5) which determines the motion of matter in dependence of the gravitational field ϕ. This equation is not relevant for the following argument.)

If the EM field couples to gravity, it must also be a source of gravity. This coupling is then described by the EMT of Maxwell theory Eq. (6.110) (in its symmetric, gauge-invariant form). The problem is that the trace of this particular EMT vanishes identically, $T_{\text{em}} = (T_{\text{em}})^\mu{}_\mu = 0$ (check this!), so that the scalar gravitational field and the EM field do not "feel" each other. In particular, there is no bending of light in the vicinity of massive bodies.

- Wrong value for perihelion precession (even with a wrong sign) ☹

  The ↑ *perihelion precession* of Mercury deviates measurably from its Newtonian value (which is caused by perturbations by other planets). For Einstein, this anomaly served as a "litmus test" on his quest to generalize SPECIAL RELATIVITY, and his first application of GENERAL RELATIVITY was the successful explanation of Mercury's anomalous perihelion precession [13] (→ *later*). Thus any theory that does not predict the correct value for the perihelion precession cannot be correct.

Historically, this first approach [Eq. (8.7)] to patch up Newton's theory and make it consistent with SPECIAL RELATIVITY goes back to the Finnish physicist GUNNAR NORDSTRÖM. He quickly dismissed Eq. (8.7) because of fundamental problems (especially its linearity, → *below*). He then proposed another (non-linear) scalar theory of gravity (↑ *Nordström's theory of gravitation*) which circumvented the most glaring issues but still failed to predict the bending of light (for the same fundamental reason sketched above) and produced the wrong value for the perihelion precession (even with a wrong sign!). Nonetheless, the theory merits consideration because it led EINSTEIN and ADRIAAN FOKKER to a groundbreaking realization [98]: Properly reformulated, the scalar field could be interpreted as a local "stretching" of the Minkowski metric. For the first time there was a clear formal link between a relativistic theory of gravity and a geometric deformation of spacetime, where the shape of the latter is determined by the distribution of mass and energy.

For a historical account of Nordström's gravity and its role in the genesis of GENERAL RELATIVITY see Refs. [99, 100].

7 | Second try: Make potential a *vector field*:

Since scalar gravity fails to match observations, a natural next step would be to consider a vector field and treat gravity analogous to Maxwell theory. This is also reasonable insofar as the gravitational potential of a point mass in Newton's theory and the Coulomb potential of a point charge in Maxwell's theory share the same form. For example, we can take Eq. (6.121) as a blueprint and propose an analogous Lagrangian for a vector gravitational field:

Eq. (6.121) → ∢ Vector field ϕ^μ & particle with trajectory $y^\mu(\lambda)$:

$$S_G[y,\phi] \overset{?}{=} \underbrace{\frac{+1}{16\pi G}\int \mathrm{d}^4x\, G_{\mu\nu}G^{\mu\nu}}_{\text{Gravitational field}} \underbrace{-mc\int\sqrt{\dot y^\mu \dot y_\mu}\,\mathrm{d}\lambda}_{\text{Relativistic particle}} \underbrace{-\frac{m}{c}\int \phi_\mu \dot y^\mu\,\mathrm{d}\lambda}_{\substack{\text{Coupling}\\ \text{with "charge" } m}} \tag{8.9}$$

with $\dot y^\mu = \frac{\mathrm{d}y^\mu}{\mathrm{d}\lambda}$ and the "gravitational field strength tensor" $G_{\mu\nu} := \partial_\mu\phi_\nu - \partial_\nu\phi_\mu$.

- Note the sign difference compared to Eq. (6.121)!
  This ensures that equal charges (= masses) *attract* each other.

- The Lagrangian for the relativistic particle differs from the one given in Exercise 7.2 in MISNER *et al.* [2]; the two are equivalent and lead to the same equations of motion.

$\xrightarrow{*}$ Results:

For details see Exercise 7.2 in MISNER *et al.* [2]; see also Ref. [97].

- No bending of light ☹
- Wrong perihelion precession ☹
- Gravitational waves have negative energy ☹

8 | Third try: Make potential a *tensor field*:

At that point, desperation starts to kick in. But since scalar and vector fields failed miserably, we have no other choice: add another index and consider a tensor field. Interestingly, this makes it rather straightforward to write down a Lorentz-covariant modification of Eq. (8.8) [or Eq. (8.4)] where we no longer must butcher the EMT by taking a trace:

$\triangleleft$ Symmetric tensor field $\phi^{\mu\nu} = \phi^{\nu\mu}$:

$$\partial^2 \phi^{\mu\nu} = -\frac{16\pi G}{c^4} T^{\mu\nu} \tag{8.10}$$

The EMT on the right is the symmetric BRT of whatever matter occupies space (Section 6.3.2).

$\xrightarrow{*}$ Results:

For details: ➔ Problemset 1 (also Exercise 7.3 in MISNER *et al.* [2])

- Light is bent around gravitational potentials ☺
- Gravitational waves have positive energy ☺
- Describes perihelion precession not correctly ☹
- Theory not self-consistent ☹

Notes:

- Eq. (8.10) will describe the *linearized* version of the correct field equations of GENERAL RELATIVITY (the → *Einstein field equations*) with $\phi^{\mu\nu}$ essentially the (small) deviation of the metric tensor from flat Minkowski space.
- That the linear tensor theory of gravity Eq. (8.10) is not self-consistent follows if one completes the theory with dynamic matter (which is the source of the gravitational field, but also influenced by the latter). Then one can show that this system of differential equations as no solution.
- As we will discuss below, the deficiency of this theory is its *linearity* (in the gravitational field); this is the root cause for its inconsistency and wrong predictions. And here comes a fascinating insight: One can show [101] that if one systematically fixes the inconsistencies of this theory, it becomes inevitably non-linear and one eventually ends up with the *correct* equations of GENERAL RELATIVITY (which we will find much later via a different route)!

9 | So far, all our tentative theories of relativistic gravity failed (none of them describe observations correctly and they even suffer from intrinsic inconsistencies). There is a simple argument why this must be so, and why the correct theory *must* be more complicated:

i | The source (= charge) of gravity is, by definition, the *gravitational mass* m_G.

This is a physically vacuous statement.

ii | A relativistic theory of gravity must be a field theory with a dynamical field.

This is necessary so that gravity does not propagate faster than the speed of light.

iii | Since the field is dynamical, it has a non-vanishing energy density.

Recall that energy is the Noether charge of time translations and therefore generates the time evolution (think of the Hamiltonian in quantum mechanics).

iv | As a relativistic theory it must obey the mass-energy equivalence: $E_0 = m_{\mathrm{I}}c^2$.

We write m_{I} to emphasize that SPECIAL RELATIVITY only knows about *inertial* mass.

v | Experiments tell us that inertial and gravitational mass are the same: $m_{\mathrm{I}} = m_{\mathrm{G}}$.

We will discuss this, and the closely related → *equivalence principle*, in detail below.

vi | Thus a gravitational field has a non-vanishing density of gravitational mass m_{G}.

Please appreciate how strange this is! If an analog statement were true for Maxwell theory (which it is not), electromagnetic waves would be *electrically charged*, and other electromagnetic waves could scatter off them!

vii | Excitations of the gravitational field are sources of the gravitational field.

This means that a relativistic theory of gravity must allow for self-interaction. In particular, it *cannot* feature a ↓ *superposition principle* and the field equations must be *non-linear*.

$\rightarrow$

The field theory of gravity must be *non-linear* and allow for self-interactions.

- All of the above theories are linear in the gravitational field; hence they are bound to fail!
- This argument also clarifies the fundamental difference between relativistic theories of *gravity* and *electrodynamics* (both of which are classical field theories that mediate forces): The EM field is also dynamical and carries energy, hence, via the mass-energy equivalence and the equivalence of inertial and heavy mass, it is a source of *gravity*. But the mass/energy carried by the EM field is not the source of the *EM field* (electrical charge is). Thus Maxwell theory does not close the "vicious circle" from above and can be both *relativistic* and *linear*.
- If you want an even more boiled down version:

  Gravity is special in RELATIVITY, because SPECIAL RELATIVITY has something to say about (inertial) mass ($E_0 = m_{\mathrm{I}}c^2$) and the latter is – via the → *equivalence principle* – the source of gravity.