↓Lecture 15 [06.02.24]

6 | Coupling to a static EM field:

The KGE can be coupled to the gauge field of electrodynamics. This is necessary to described charged particles (in particular: the hydrogen atom). Note that in the following the gauge field is a *parameter* and not a dynamic degree of freedom.

i | Goal: Construct Lagrangian density that is …

- … a Lorentz scalar.
- … quadratic in ϕ.
- … gauge invariant under the gauge transformation $A'_\mu = A_\mu - \partial_\mu \lambda$.
- … couples ϕ and A_μ in a non-trivial way.

Without additional tools, this is a tough job!

ii | ◁ Gauge transformation $A'_\mu = A_\mu - \partial_\mu \lambda(x)$

Let us assume that the KG field transforms under the gauge transformation as follows:

$$\phi'(x) := e^{iQ\lambda(x)}\phi(x) \quad \text{with the ⁂ } U(1) \textit{ charge } Q = \frac{q}{\hbar c} \in \mathbb{R}\,. \tag{7.36}$$

q: electric charge of the particle described by the wavefunction ϕ

- It is reasonable to assume that the KG field must transform via phase factors because we already know [recall Eq. (7.19)] that the KG Lagrangian is invariant under *global* phase transformations $\lambda(x) = \text{const.}$ Our hope is that we can "extend" this symmetry for arbitrary non-constant $\lambda(x)$.
- The charge Q is a property of the field and quantifies how it transforms under gauge transformations; it essentially plays the role of the electric charge of the particle described by ϕ; e.g., for an electron we would set $Q = \frac{-e}{\hbar c} < 0$.

  The additional division by $\hbar c$ is necessary for dimensional reasons: $[\lambda] = L[\varphi]$ with $A^\mu = (\varphi, \vec{A})$; therefore $[\lambda q] = L[\varphi q] = L[E] = \frac{ML^3}{T^2}$ and it is $[\hbar c] = \frac{ML^3}{T^2}$ as well. In natural units (where $\hbar = 1 = c$, $Q = q$ is simply the electric charge.
- The term "$U(1)$ charge" highlights that the gauge transformation $e^{iQ\lambda(x)} \in \mathrm{U}(1)$ is a $U(1)$ gauge transformation; the charge is the generator of this Lie group.

iii | Problem:

Derivatives transform complicated under gauge transformations:

$$\partial_\mu \phi'(x) \mathrel{\hat{=}} e^{iQ\lambda(x)}\left[iQ(\partial_\mu\lambda)\phi(x) + \partial_\mu\phi(x)\right] \tag{7.37}$$

→ It is hard to combine derivatives of fields to construct gauge-invariant terms!

Solution:

Define the …

$$\boxed{\text{⁂ } \textit{(Gauge) Covariant derivative:} \quad D_\mu := \partial_\mu + iQA_\mu} \tag{7.38}$$

→ Lorentz vector (thus we can it use to construct Lorentz scalars!)

The covariant derivative has the following useful property:

$$D'_\mu \phi' = \left[\partial_\mu + iQA_\mu - iQ(\partial_\mu \lambda)\right] e^{iQ\lambda}\phi \mathrel{\hat{=}} e^{iQ\lambda} D_\mu \phi \tag{7.39}$$

$\rightarrow D_\mu \phi$ transforms like ϕ under gauge transformations. [and not as ugly as Eq. (7.37)!]

This is useful because it allows us to combine derivatives into gauge-invariant terms.

iv | Using the covariant derivative, we can now construct the following general Lagrangian density that satisfies our four requirements above:

$$\boxed{\mathcal{L}_A(\phi, \partial\phi) = (D^\mu \phi)(D_\mu \phi)^* - M^2 \phi\phi^*} \tag{7.40}$$

Please appreciate the ingenuity of the term $(D^\mu\phi)(D_\mu\phi)^*$: It is *Lorentz invariant* because we pair the indices correctly, and it is *gauge invariant* because we pair $(D^\mu\phi)$ with its complex conjugate $(D_\mu\phi)^*$ (which is sufficient because $D^\mu\phi$ gauge-transforms like ϕ).

This Lagrangian density is gauge-invariant by construction in the sense that

$$\mathcal{L}_A(\phi, \partial\phi) = \mathcal{L}_{A'}(\phi', \partial\phi') \quad \text{or} \quad \mathcal{L}(\phi, D\phi) = \mathcal{L}(\phi', D'\phi')\,. \tag{7.41}$$

- A comparison of the free Klein-Gordon Lagrangian Eq. (7.11) and the new one Eq. (7.40) reveals that we simply made the substitution $\partial_\mu \mapsto D_\mu$, i.e., we replaced partial derivatives by covariant derivatives (which depend on the gauge field). This trick is not specific to the Klein-Gordon field and yields gauge-invariant theories in general. This procedure is called ↑ *minimal coupling*.
- Note that the transformation Eq. (7.36) is a *local* phase rotation of the KG-field. In Eq. (7.17) we considered a *global* phase rotation and identified it as a continuous symmetry of the KG Lagrangian $\mathcal{L}_{\text{KG}}$. You can check that the new *local* transformation does not leave $\mathcal{L}_{\text{KG}}$ invariant, but it does leave $\mathcal{L}_A$ invariant if A^μ transforms together with ϕ as defined above. The transition from $\mathcal{L}_{\text{KG}}$ (with a global symmetry) to $\mathcal{L}_A$ (with a local version of the same symmetry) is called *gauging the symmetry*. You can use this line of reasoning to "invent" the electromagnetic gauge field: If you start from a global continuous symmetry and demand that it becomes a local symmetry, you have to pay for it by introducing a new field: the gauge field.

v | Klein-Gordon equation in a static EM field:

The Euler-Lagrange equations of $\mathcal{L}_A$ yield: Eq. (6.6) $\xrightarrow{\text{Eq. (7.40)}}$

$$\boxed{(D^2 + M^2)\phi(x) = 0} \tag{7.42}$$

with $D^2 = D_\mu D^\mu$ and $M = \frac{mc}{\hbar}$.

In the form Eq. (7.42) both Lorentz covariance and gauge invariance are manifest (because we use the covariant derivative). If we expand everything, we loose these features but obtain a less abstract (but more complicated) form of the PDE:

$$\left[\frac{1}{c^2}\left(\partial_t + iQc\varphi\right)^2 - \left(\nabla - iQ\vec{A}\right)^2 + \frac{m^2c^2}{\hbar^2}\right]\phi(t, \vec{x}) = 0 \tag{7.43}$$

Here we used $A_\mu = (\varphi, -\vec{A})$ (covariant!).

vi | Example: Hydrogen atom

Goal: Describe the electron of the hydrogen atom in the static EM field generated by the proton in terms of the KGE; i.e., we interpret the KG field ϕ naïvely as the wavefunction of the electron. Our hope is that the energy spectrum of this relativistic theory explains the observed fine-structure splitting.

a | $\lhd$ Coulomb potential (of proton with charge $e > 0$) $\rightarrow$

$$\text{Choose a gauge where} \quad \varphi(x) = \frac{e}{|\vec{x}|} \quad \text{and} \quad \vec{A} = \vec{0} \tag{7.44}$$

$\overset{\circ}{\rightarrow}$ With electron charge $Q = \frac{-e}{\hbar c} < 0$ one finds:

$$\left[\frac{1}{c^2}\left(i\,\partial_t + \frac{e^2}{\hbar|\vec{x}|}\right)^2 + \nabla^2 - \frac{m^2c^2}{\hbar^2}\right]\phi(t,\vec{x}) = 0 \tag{7.45}$$

b | $\lhd$ Ansatz $\phi(t,\vec{x}) = \tilde{\phi}(\vec{x})e^{-\frac{i}{\hbar}Et}$ $\rightarrow$ "Stationary" Klein-Gordon equation:

$$\left[c^2\hbar^2\Delta + \left(E + \frac{e^2}{|\vec{x}|}\right)^2 - m^2c^4\right]\tilde{\phi}(\vec{x}) = 0 \tag{7.46}$$

Note that this PDE is *quadratic* in the energy E (and not linear, like the time-independent Schrödinger equation).

c | One can use a clever mapping to the non-relativistic Schrödinger equation to solve for $\tilde{\phi}(\vec{x})$ and determine the energies E for which solutions exist:

$$\overset{*}{\rightarrow} \quad E_{n,l} = \frac{mc^2}{\sqrt{1 + \frac{\alpha^2}{(n-\delta_l)^2}}} \quad \text{with} \quad \delta_l = l + \tfrac{1}{2} - \sqrt{\left(l + \tfrac{1}{2}\right)^2 - \alpha^2}\,. \tag{7.47}$$

Here $n = 1, 2, \ldots$ is the ↓ *principal quantum number* and $l = 0, 1, 2, \ldots$ is the ↓ *orbital angular momentum quantum number*. $\alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137}$ is the fine-structure constant.

d | Comments:

- The spectrum Eq. (7.47) predicts a splitting of the l-degeneracy; recall that this degeneracy is perfect in the non-relativistic hydrogen atom [cf. Eq. (7.8)]. Unfortunately, the spectrum Eq. (7.47) does *not* match observations! The reason is that the Klein-Gordon equation does not know about the *electron spin*. Schrödinger and his contemporaries were aware of this solution and its problems (this shines through in the quotes at the beginning of this chapter). This failure to predict the fine-structure correctly led to the dismissal of the Klein-Gordon equation and motivated Paul Dirac to search for another equation (→ *next section*).
- Today we know that the Klein-Gordon equation is *not wrong*: It simply does not apply to particles with non-zero spin (and the electron in the hydrogen atom happens to have spin $s = \frac{1}{2}$). However, it *does* apply to spin-0 particles like ↑ *kaons* (K mesons, bound states of two quarks), ↑ *pions* (pi mesons), and the ↑ *Higgs boson* (the latter being the only *elementary* particle with zero spin). But since we cannot build hydrogen atoms out of these particles, the significance of the above solution remains limited.

7 | First-order formulation:

Here we consider again the free KGE (without EM field) for simplicity.

i | KGE = *Second-order* PDE in time

Problem: $\phi(t = 0, \vec{x})$ does *not* specify the state of the system completely [unlike for the Schrödinger equation one also needs $\dot{\phi}(t = 0, \vec{x})$ to pick out a unique solution $\phi(t, \vec{x})$].

Recall: Every higher-order differential equation can be recast as a first-order differential equation with multiple components.

→ Goal: Rewrite the KGE in the first-order form

$$i\hbar\partial_t \Phi = \hat{H}_{\mathrm{KG}}\Phi \quad \text{with} \quad \Phi = \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix} . \tag{7.48}$$

Downside: In this form, the KGE is no longer manifest Lorentz covariant.

ii | Define

$$\phi_\pm := \frac{1}{2}\left(\phi \pm \frac{i\hbar}{mc^2}\partial_t\phi\right) \tag{7.49}$$

so that

$$\phi = \phi_+ + \phi_- \quad \text{and} \quad \frac{i\hbar}{mc^2}\partial_t\phi = \phi_+ - \phi_- . \tag{7.50}$$

iii | Define the 2×2 differential operator

$$\hat{H}_{\mathrm{KG}} := \begin{pmatrix} \hat{H}_0 + mc^2 & \hat{H}_0 \\ -\hat{H}_0 & -\hat{H}_0 - mc^2 \end{pmatrix} = \hat{H}_0 \otimes (\sigma^z + i\sigma^y) + mc^2\sigma^z \tag{7.51}$$

with $\hat{H}_0 = -\frac{\hbar^2}{2m}\nabla^2$ the free particle Hamiltonian and the Pauli matrices

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{7.52}$$

$\hat{H}_{\mathrm{KG}}$ is a linear operator on the Hilbert space $L^2 \otimes \mathbb{C}^2$ of two-component square-integrable functions. Note that $\hat{H}_{\mathrm{KG}}^\dagger = \hat{H}_0 \otimes (\sigma^z - i\sigma^y) + mc^2\sigma^z \neq \hat{H}_{\mathrm{KG}}$ is non-Hermitian with respect to the conventional inner product on $L^2 \otimes \mathbb{C}^2$:

$$\langle\Phi|\Psi\rangle_{L^2\otimes\mathbb{C}^2} = \int \mathrm{d}^3x\, \Phi^\dagger(x)\Psi(x) = \int \mathrm{d}^3x\, \left(\phi_+^*\psi_+ + \phi_-^*\psi_-\right) . \tag{7.53}$$

iv | Check that the differential equation in first-order Schrödinger form

$$i\hbar\partial_t\Phi = \hat{H}_{\mathrm{KG}}\Phi \quad \Leftrightarrow \quad \begin{cases} i\hbar\partial_t\phi_+ = (\hat{H}_0 + mc^2)\phi_+ + \hat{H}_0\phi_- \\ i\hbar\partial_t\phi_- = -\hat{H}_0\phi_+ - (\hat{H}_0 + mc^2)\phi_- \end{cases} \tag{7.54}$$

is equivalent to the KGE:

a | Indeed, the *difference* of the two equations yields

$$-\frac{\hbar^2}{mc^2}\partial_t\chi = (\hat{H}_0 + mc^2)\phi + \hat{H}_0\phi \quad \Leftrightarrow \quad \frac{1}{c^2}\partial_t\chi - \nabla^2\phi + \frac{m^2c^2}{\hbar^2}\phi = 0 \tag{7.55}$$

where we defined $\phi := \phi_+ + \phi_-$ and $\chi := \frac{mc^2}{i\hbar}(\phi_+ - \phi_-)$.

b | By contrast, the *sum* of the two equation yields

$$mc^2\partial_t\phi = (\hat{H}_0 + mc^2)\chi - \hat{H}_0\chi \quad \Leftrightarrow \quad \partial_t\phi = \chi . \tag{7.56}$$

c | Combining Eq. (7.55) and Eq. (7.56) returns the KGE:

$$\frac{1}{c^2}\partial_t^2\phi - \nabla^2\phi + \frac{m^2c^2}{\hbar^2}\phi = 0 . \tag{7.57}$$

v | If one defines the

$$\text{⁂ Klein-Gordon adjoint} \quad \bar{\Phi} := \Phi^\dagger \sigma^z = \left(\phi_+^*, -\phi_-^*\right) , \tag{7.58}$$

one can express the Klein-Gordon sesquilinear form Eq. (7.25) as

$$\langle \Phi | \Psi \rangle_{\mathrm{KG}} := \int \mathrm{d}^3x \, \bar{\Phi}(x) \Psi(x) \overset{7.49}{=} \frac{i\hbar}{2mc^2} \int \mathrm{d}^3x \, \left(\phi^* \dot{\psi} - \dot{\phi}^* \psi\right) \overset{7.25}{=} \langle \phi | \psi \rangle_{\mathrm{KG}} . \tag{7.59}$$

Remember that this is not a proper inner product because it is not positive-definite.

vi | If one defines additionally for an operator A on $L^2 \otimes \mathbb{C}^2$ the

$$\text{⁂ Klein-Gordon adjoint} \quad \bar{A} := \sigma^z A^\dagger \sigma^z , \tag{7.60}$$

it follows $\overline{A\Phi} = \bar{\Phi}\bar{A}$ and $\bar{\bar{A}} = A$, and thereby

$$\langle \Phi | A \Psi \rangle \overset{7.59}{=} \langle \bar{A}\Phi | \Psi \rangle . \tag{7.61}$$

vii | It is easy to verify that the KG Hamiltonian is "Klein-Gordon Hermitian", namely

$$\bar{\hat{H}}_{\mathrm{KG}} \overset{7.51}{=} \hat{H}_{\mathrm{KG}} \tag{7.62}$$

because $\sigma^z \sigma^y \sigma^z = -\sigma^y$.

viii | With this machinery, we have now a new method to check that the time-evolution generated by the KGE leaves the KG sesquilinear form invariant:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \langle \phi | \psi \rangle_{\mathrm{KG}} &\overset{7.59}{=} \frac{\mathrm{d}}{\mathrm{d}t} \langle \Phi | \Psi \rangle_{\mathrm{KG}} && (7.63a)\\
&= \langle \Phi | \dot{\Psi} \rangle_{\mathrm{KG}} + \langle \dot{\Phi} | \Psi \rangle_{\mathrm{KG}} && (7.63b)\\
&\overset{7.54}{=} \frac{1}{i\hbar} \langle \Phi | \hat{H}_{\mathrm{KG}} \Psi \rangle_{\mathrm{KG}} - \frac{1}{i\hbar} \langle \hat{H}_{\mathrm{KG}} \Phi | \Psi \rangle_{\mathrm{KG}} && (7.63c)\\
&\overset{7.61}{\underset{7.62}{=}} \frac{1}{i\hbar} \left(\langle \Phi | \hat{H}_{\mathrm{KG}} \Psi \rangle_{\mathrm{KG}} - \langle \Phi | \hat{H}_{\mathrm{KG}} \Psi \rangle_{\mathrm{KG}} \right) = 0 && (7.63d)
\end{aligned}$$

We already knew this from Noether's theorem, but it is always nice to derive such statements in various ways.

8 | Non-relativistic limit:

i | Goal: Derive a non-relativistic approximation of the Klein-Gordon equation

$$\left[\frac{1}{c^2} \partial_t^2 - \nabla^2 + \frac{m^2 c^2}{\hbar^2} \right] \phi(t, \vec{x}) = 0 . \tag{7.64}$$

ii | ◁ Kinetic energy: $E_{\mathrm{kin}} = E - mc^2 = \sqrt{\vec{p}^2 c^2 + m^2 c^4} - mc^2 \approx \frac{1}{2} m v^2 + \mathcal{O}(\beta^4)$
(Note that both E_{kin} and E are non-negative!)
◁ Ansatz:

$$\phi_\pm(t, \vec{x}) = \tilde{\phi}_\pm(\vec{x}) e^{\mp \frac{i}{\hbar} E t} = \underbrace{\tilde{\phi}_\pm(\vec{x}) e^{\mp \frac{i}{\hbar} E_{\mathrm{kin}} t}}_{=:\hat{\phi}_\pm(t,\vec{x})} e^{\mp \frac{i}{\hbar} mc^2 t} \tag{7.65}$$

$\hat{\phi}(t, \vec{x})$ contains only the time evolution due to the kinetic energy, excluding the rest energy.

iii | If we use that

$$\partial_t^2 \hat{\phi}_\pm = -\frac{E_{\text{kin}}^2}{\hbar^2}\hat{\phi}\,, \tag{7.66}$$

we can make the following approximation in the non-relativistic limit $E_{\text{kin}} \ll mc^2$:

$$\partial_t^2 \phi_\pm = e^{\mp\frac{i}{\hbar}mc^2 t}\left\{\partial_t^2\hat{\phi}_\pm \mp \frac{2imc^2}{\hbar}\partial_t\hat{\phi}_\pm - \left(\frac{mc^2}{\hbar}\right)^2\hat{\phi}_\pm\right\} \tag{7.67a}$$

$$= -e^{\mp\frac{i}{\hbar}mc^2 t}\left\{\pm\frac{2imc^2}{\hbar}\partial_t\hat{\phi}_\pm + \left(\frac{mc^2}{\hbar}\right)^2\left[1+\left(\frac{E_{\text{kin}}}{mc^2}\right)^2\right]\hat{\phi}_\pm\right\} \tag{7.67b}$$

$$\approx -e^{\mp\frac{i}{\hbar}mc^2 t}\left\{\pm\frac{2imc^2}{\hbar}\partial_t\hat{\phi}_\pm + \left(\frac{mc^2}{\hbar}\right)^2\hat{\phi}_\pm\right\} \tag{7.67c}$$

iv | Eq. (7.67c) in Eq. (7.64) yields:

$$e^{\mp\frac{i}{\hbar}mc^2 t}\left[\pm\frac{2im}{\hbar}\partial_t + \frac{m^2c^2}{\hbar^2} + \nabla^2 - \frac{m^2c^2}{\hbar^2}\right]\hat{\phi}_\pm(t,\vec{x}) = 0 \tag{7.68}$$

And finally:

$$\boxed{\pm i\hbar\partial_t\hat{\phi}_\pm(t,\vec{x}) = -\frac{\hbar^2}{2m}\nabla^2\hat{\phi}_\pm(t,\vec{x})} \tag{7.69}$$

This is the Schrödinger equation for a free particle.

Note that the "negative energy solutions" ϕ_- lead to the *time-inverted* Schrödinger equation.

7.2. The Dirac equation

The Dirac equation was published by Paul Dirac in [87], only two years after Schrödinger published the Schrödinger equation.

1 | Goal:

The Klein-Gordon equation has a few undesirable quirks:

- *It's conserved* U(1) *current has no positive-definite density and therefore cannot be interpreted as a probability current. Conversely, the conventional norm on L^2 is not conserved. In the first-order formulation, this corresponds to a non-Hermitian Hamiltonian.*

  → Can we construct a relativistic field equation with a conserved positive-definite density that gives rise to a norm and a Hermitian Hamiltonian?

- *In its manifest Lorentz covariant formulation, the KGE is of second order in time, so that we must provide both the wavefunction and its time derivative as initial data.*

  → Can we construct a relativistic field equation which is first order in time (just like the Schrödinger equation)?

- *For each momentum there is are two solutions: one with positive and one with negative energy.*

  → Can we get rid of the negative energy solutions?

The Dirac equation succeeds in solving the first two issues – but not the last one, i.e., there will still be negative energy solutions.

2 | Observation:

To reach our goals we must equip our "toolbox" of tensor calculus with additional building blocks. As it turns out, there is another type of field (besides the tensor fields we introduced in Chapter 3) that plays an important role in quantum mechanics: ↑ *spinor fields*.

Remember: Vector fields under rotations: $\vec{\phi}'(\vec{x}) = R\vec{\phi}(R^{-1}\vec{x})$

→ In general, a field $\phi(x) \in \mathbb{C}^n$ can transform under homogeneous Lorentz transformations as

$$\phi'_a(x) = M_{ab}(\Lambda)\phi_b(\Lambda^{-1}x) \qquad a = 1, \dots, n \tag{7.70}$$

where

$$M(\Lambda')M(\Lambda)\phi(\Lambda^{-1}\Lambda'^{-1}x) \overset{!}{=} M(\Lambda'\Lambda)\phi((\Lambda'\Lambda)^{-1}x) \tag{7.71}$$

is a n-dimensional representation of the (proper orthochronous) Lorentz group $\mathrm{SO}^+(1,3)$.

- Regarding groups and their representations: ➲ Problemset 1.
- More explicitly: The tensor fields (of various rank) we know so far allow only for the description of particles with *integer spin* $S = 0, 1, 2, \cdots$ (spin = internal angular momentum). What we are missing are fields that can describe particles with *half-integer spin* $S = \frac{1}{2}, \frac{3}{2}, \cdots$; these are the spinor fields.

  The reason why this is crucial for relativistic *quantum mechanics* in particular has to do with the fact that multiplying wave functions by a global phase does not change the state. In mathematical parlance we are dealing with ↑ *projective Hilbert spaces* and ↑ *projective representations* of symmetries. Thus if you are interested what rotations SO(3) do to the quantum state of your system, you must study all *projective* representations of SO(3). It turns out that these can be identified with the "conventional" (= linear) representations of another group: SU(2) (the so called ↑ *double cover* of SO(3)). And you know that the irreducible representations of SU(2) are labeled by "spin quantum numbers" $s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$. In general, the double covers of $\mathrm{SO}(n)$ are called ↑ *spin groups* Spin(n), and similarly, the double cover of the proper orthochronous Lorentz group $\mathrm{SO}^+(1,3)$ is the group $\mathrm{Spin}(1,3) \simeq \mathrm{SL}(2,\mathbb{C})$ (the group of complex 2×2 matrices with determinant one). It turns out that the irreducible representations of this group can be labeled by *two* numbers (m, n) with $m, n = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$. The spinor representations we are interested in (the ones missing from our discussion of tensor fields) are the ones for which $m + n$ is half-integer. Conversely, the $(\frac{1}{2}, \frac{1}{2})$ representation is our well-known 4-vector representation A^μ and the $(0, 0)$ representation is that of a scalar like ϕ.

3 | We want a first-order relativistic field equation → Ansatz:

$$(\partial^\mu \partial_\mu + \text{const})\phi = 0 \quad \Rightarrow \quad (i\blacksquare^\mu \partial_\mu + \text{const})\phi = 0 \tag{7.72}$$

We do not yet know what ■ is (only that it cannot be a derivative).
The i anticipates wave-like solutions for real ■.

A covariant equation of the form $\partial_\mu \phi = 0$ or $\partial_\mu A^\mu = 0$ would of course also be possible; their solutions, however, are either too simple or do not match observations.

4 | Then (combine 2 & 3)

i | ◁ Coordinate transformation $x' = \Lambda x$ & Field transformation $\phi'(x') = M(\Lambda)\phi(x)$

ii | ◁ ϕ with $(i\blacksquare^\mu \partial_\mu + \text{const})\phi(x) = 0$ for all x

That is, $\phi(x)$ is a solution of the equation we want to construct.

iii | When is $\phi'(x) = M(\Lambda)\phi(\Lambda^{-1}x)$ is a new solution?

We want the equation to be Lorentz covariant; this means that the Lorentz group must be (part of) its invariance group: Lorentz transformations map solutions to new solutions.

$$(i\blacksquare^\mu\partial_\mu + \text{const})\phi'(x) = [i\blacksquare^\mu(\Lambda^{-1})^\nu{}_\mu\partial_\nu + \text{const}]\, M(\Lambda)\phi(\Lambda^{-1}x) \overset{!}{=} 0 \tag{7.73}$$

Multiply with $M^{-1}(\Lambda)$:

$$\Leftrightarrow \quad [i\underbrace{M^{-1}(\Lambda)\blacksquare^\mu M(\Lambda)(\Lambda^{-1})^\nu{}_\mu}_{\overset{!}{=}\blacksquare^\nu}\partial_\nu + \text{const}]\,\phi(\Lambda^{-1}x) \overset{!}{=} 0 \tag{7.74}$$

$\rightarrow \blacksquare^\mu \equiv \gamma^\mu$ must be $n \times n$-matrices with

$$\boxed{M^{-1}(\Lambda)\gamma^\mu M(\Lambda) = \Lambda^\mu{}_\nu\gamma^\nu} \tag{7.75}$$

The γ-matrices "translate" the "spinor"-representation $M(\Lambda)$ into the "vector"-representation Λ and vice versa.

5 | *Question:* How to find appropriate γ^μ and $M(\Lambda)$ that satisfy Eq. (7.75)?

Remember: $\mathrm{SO}^+(1,3)$ is a Lie group (Recall ➲ Problemset 4):

$$\Lambda = \exp\left[-\frac{i}{2}\omega_{\alpha\beta}\mathcal{J}^{\alpha\beta}\right] \overset{\omega\ll 1}{\approx} \mathbb{1} - \frac{i}{2}\omega_{\alpha\beta}\mathcal{J}^{\alpha\beta} \tag{7.76a}$$

$$M(\Lambda) = \exp\left[-\frac{i}{2}\omega_{\alpha\beta}\mathcal{S}^{\alpha\beta}\right] \overset{\omega\ll 1}{\approx} \mathbb{1} - \frac{i}{2}\omega_{\alpha\beta}\mathcal{S}^{\alpha\beta} \tag{7.76b}$$

$\omega_{\alpha\beta}$ antisymmetric tensor $\rightarrow$ 3 rotations (angles) + 3 boosts (rapidities)

It is $(\mathcal{J}^{\alpha\beta})_{\mu\nu} = i(\delta^\alpha_\mu\delta^\beta_\nu - \delta^\alpha_\nu\delta^\beta_\mu)$; these 4×4 matrices $\mathcal{J}^{\alpha\beta}$ generate the 4-vector representation $(\frac{1}{2},\frac{1}{2})$, i.e., the 4×4-matrices Λ. By contrast, the $n\times n$-matrices $\mathcal{S}^{\alpha\beta}$ generate the spinor-representation $M(\Lambda)$ [we will find $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$]. The generators are antisymmetric in the spacetime indices.

- Infinitesimal form of Eq. (7.75):

$$\left[\gamma^\mu, \mathcal{S}^{\alpha\beta}\right] \mathrel{\hat{=}} (\mathcal{J}^{\alpha\beta})^\mu{}_\nu\gamma^\nu \mathrel{\hat{=}} i(\eta^{\alpha\mu}\gamma^\beta - \eta^{\beta\mu}\gamma^\alpha) \tag{7.77}$$

- $\lhd\ \mathcal{J}^{\alpha\beta}$ (➲ Problemset 4) $\rightarrow$ Lie-algebra of Lorentz group ($J = \mathcal{S}, \mathcal{J}$):

$$\left[J^{\mu\nu}, J^{\rho\sigma}\right] \mathrel{\hat{=}} i(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\mu\sigma}J^{\nu\rho}) \tag{7.78}$$

The Lie algebra defines the structure of the Lie group by exponentiation and is therefore the same for all representations, recall Eq. (4.63).

6 | Solution to Eq. (7.75) via Dirac's trick [87]: $\lhd\ \gamma^\mu$ such that

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\,\mathbb{1}_{n\times n} \quad \text{⁂ } \textit{Dirac algebra}} \tag{7.79}$$

with the ↓ *anticommutator* $\{X, Y\} = XY + YX$.

- Matrices $\gamma^\mu = (\gamma^0, \gamma^1, \gamma^2, \gamma^3)$ that satisfy Eq. (7.79) are called ⁂ *Dirac matrices* or ⁂ *Gamma matrices*.
- This is the 16-dimensional *Clifford algebra* $\mathrm{C}\ell_{1,3}(\mathbb{C})$.

Then

$$\boxed{\mathcal{S}^{\mu\nu} := \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right]} \tag{7.80}$$

satisfies the Lorentz algebra Eq. (7.78) and Eq. (7.77).

Check this by plugging Eq. (7.80) into Eq. (7.78) and Eq. (7.77) and using Eq. (7.79)!

→ Problem of solving Eq. (7.75) has been reduced to finding 4 matrices γ^μ that satisfy Eq. (7.79).

7 | Representations of Eq. (7.79):

- At least $n = 4$-dimensional
  (Think of the γ^μ as Majorana modes and construct ladder operators → 2 modes.)
- All 4-dimensional representations are unitarily equivalent
  (Actually, they constitute the *unique* irrep of the Dirac algebra which is 4-dimensional.)
- We use the Weyl representation (sometimes called *chiral representation*):

$$\boxed{\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \quad \text{and} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \quad i = 1, 2, 3} \tag{7.81}$$

  - Recall the Pauli matrices Eq. (7.52):

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7.82}$$

  - Other common choices are the ↑ *Dirac representation* and the ↑ *Majorana representation*.
- Henceforth: $\Lambda_{\frac{1}{2}} \equiv M(\Lambda)$
  It turns out that these are two "copies" of a spin-$\frac{1}{2}$ projective representation: $\Lambda_{\frac{1}{2}}$ corresponds to the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation of $\mathrm{SL}(2, \mathbb{C})$. Since $n + m = \frac{1}{2}$, this is a spinor representation, i.e., a projective representation of the Lorentz group $\mathrm{SO}^+(1, 3)$. The fact that it is the sum of two such representations makes it *reducible*. The wavefunction $\Psi(x)$ has therefore $n = 4$ components and is a *spinor field* (and not a tensor field).

8 | Setting const $= -M = -\frac{mc}{\hbar}$ (which has dimension of an inverse length), we find:

$$\boxed{(i\gamma^\mu \partial_\mu - M)\Psi = 0 \quad ⁂ \textit{ Dirac equation}} \tag{7.83}$$

Here, $\Psi(x)$ is a ⁂ *(bi)spinor*-field:

$$\Psi : \mathbb{R}^{1,3} \to \mathbb{C}^4 = \mathbb{C}^2 \oplus \mathbb{C}^2 . \tag{7.84}$$

Introduce the ⁂ *Feynman slash notation*: $\not{O} := \gamma^\mu O_\mu$

(Here, O_μ stands for any object with a 4-vector index.)

With the slash notation, the Dirac equation can be written as:

$$\boxed{(i\not{\partial} - M)\Psi = 0} \tag{7.85}$$

The Dirac equation is engraved in a plaque on the floor of Westminster Abbey next to Isaac Newton's tomb (they abbreviate $\gamma \cdot \partial = \gamma^\mu \partial_\mu$ and are in natural units $\hbar = 1 = c$ where $M = m$):

(Photograph from https://cerncourier.com/a/paul-dirac-a-genius-in-the-history-of-physics.)

9 | The components $\Psi_a(x)$ $(a = 1, 2, 3, 4)$ satisfy the KGE:

$$0 = (-i\gamma^\mu \partial_\mu - M)(i\gamma^\nu \partial_\nu - M)\Psi \overset{7.79}{=} (\partial^2 + M^2)\Psi \tag{7.86}$$

On the right hand side of Eq. (7.86) there is an identity $\mathbb{1}_{4\times 4}$ that we omit.

- The Dirac differential operator is the "square root" of the Klein-Gordon differential operator.
- ¡! Although Ψ has as many components as the EM gauge field A^μ, we do *not* write these components as Ψ^μ, but either simply as Ψ (and think of it as a four-dimensional column vector), or as Ψ_a with *spinor index* $a = 1, 2, 3, 4$. The purpose of this notational difference is to denote the different ways the fields transform under Lorentz transformations:

$$A'^\mu = \Lambda^\mu{}_\nu A^\nu \quad \text{versus} \quad \Psi'_a = (\Lambda_{\frac{1}{2}})_{ab}\Psi_b \quad \text{or simply} \quad \Psi' = \Lambda_{\frac{1}{2}}\Psi\,. \tag{7.87}$$

  Note that $\Lambda \equiv \Lambda^\mu{}_\nu$ and $\Lambda_{\frac{1}{2}} = M(\Lambda)$ are *not* the same 4×4 matrices!

10 | Dirac adjoint:

We would like to find a Lagrangian density for the Dirac equation; since this must be a Lorentz scalar, we ask the question:

How to form Lorentz scalars from Dirac spinors?

i | First try: $\Psi^\dagger\Psi$

$$\Psi'^\dagger\Psi' = \Psi^\dagger \underbrace{\Lambda^\dagger_{\frac{1}{2}}\Lambda_{\frac{1}{2}}}_{\neq \mathbb{1}} \Psi \neq \Psi^\dagger\Psi \tag{7.88}$$

$\Lambda_{\frac{1}{2}}$ is *not* unitary because $\mathcal{S}^{\mu\nu}$ is not Hermitian for boosts ($\mu = 0$ and $\nu = 1, 2, 3$).

This is a consequence of the ↑ *non-compactness* of the Lorentz group due to boosts.

ii | Define

$$\boxed{\bar{\Psi} := \Psi^\dagger\gamma^0 \quad \text{⁂ } \textit{Dirac adjoint}} \tag{7.89}$$

$\overset{\circ}{\rightarrow} \bar{\Psi}'\Psi' = \bar{\Psi}\Lambda_{\frac{1}{2}}^{-1}\Lambda_{\frac{1}{2}}\Psi = \bar{\Psi}\Psi \Rightarrow$ Lorentz scalar

Use Eq. (7.80) and Eq. (7.76b) and the Dirac algebra to show this!

11 | Lagrangian:

With these tools, it is reasonable to propose the following Lagrangian density:

$$\boxed{\mathcal{L}_{\text{Dirac}} = \bar{\Psi}(i\gamma^\mu\partial_\mu - M)\Psi = \bar{\Psi}(i\not{\partial} - M)\Psi} \tag{7.90}$$

$\overset{\circ}{\rightarrow}$ Euler-Lagrange equations = Dirac equation

- Note that in explicit index notation, the Lagrangian density reads

$$\mathcal{L}_{\text{Dirac}} = i\bar{\Psi}_a\gamma^\mu_{ab}(\partial_\mu\Psi_b) - M\bar{\Psi}_a\Psi_a \tag{7.91}$$

  where sums over pairs of spinor indices are implied.

  The Euler-Lagrange equations follow again by treating Ψ_a and $\bar{\Psi}_a$ as independent fields:

$$0 \overset{!}{=} \frac{\partial\mathcal{L}_{\text{Dirac}}}{\partial\bar{\Psi}_a} - 0 \qquad = i\gamma^\mu_{ab}(\partial_\mu\Psi_b) - M\Psi_a \quad = \left[(i\not{\partial} - M)\Psi\right]_a \tag{7.92a}$$

$$0 \overset{!}{=} \frac{\partial\mathcal{L}_{\text{Dirac}}}{\partial\Psi_a} - \partial_\mu\frac{\partial\mathcal{L}_{\text{Dirac}}}{\partial(\partial_\mu\Psi_a)} = -M\bar{\Psi}_a - i(\partial_\mu\bar{\Psi}_b)\gamma^\mu_{ba} \overset{\circ}{=} \left[\overline{(i\not{\partial} - M)\Psi}\right]_a \tag{7.92b}$$

  Note that the two equations are Dirac adjoints of each other.

- Let us check that $\mathcal{L}_{\text{Dirac}}$ is a Lorentz scalar:

$$\begin{aligned}
\mathcal{L}'_{\text{Dirac}} &= \bar{\Psi}'\left(i\gamma^\mu\partial'_\mu - M\right)\Psi' && (7.93a)\\
&= \bar{\Psi}\Lambda_{\frac{1}{2}}^{-1}\left(i\gamma^\mu\Lambda_\mu{}^\nu\partial_\nu - M\right)\Lambda_{\frac{1}{2}}\Psi && (7.93b)\\
&= \bar{\Psi}\left(i\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}}\Lambda_\mu{}^\nu\partial_\nu - M\right)\Psi && (7.93c)\\
&\overset{7.75}{=} \bar{\Psi}\left(i\Lambda^\mu{}_\rho\gamma^\rho\Lambda_\mu{}^\nu\partial_\nu - M\right)\Psi && (7.93d)\\
&= \bar{\Psi}\left(i\gamma^\nu\partial_\nu - M\right)\Psi = \mathcal{L}_{\text{Dirac}} && (7.93e)
\end{aligned}$$

Here we used the following fact:

$$\boxed{\text{The gamma matrices transform } \textit{not} \text{ like Lorentz vectors: } \gamma'^\mu = \gamma^\mu.} \tag{7.94}$$

This is good because otherwise the Dirac equation would be different in different inertial systems.

This also means that slashed quantities (like $\not{\partial} = \gamma^\mu\partial_\mu$) are *not* Lorentz scalars. Think of it like this: they do *not* have a Lorentz index, but they *do* have a two spinor indices (which we don't write) because they are *matrices*. To get rid of these indices, you must pair them with the indices of spinor fields. That is, slashed quantities become Lorentz scalars if put between two Dirac spinors like in the Dirac Lagrangian: $\bar{\Psi}\not{\partial}\Psi$ *is* a scalar field.

12 | Conserved current:

Now that we have a Lagrangian, it is just a straightforward application of Noether's theorem to obtain the conserved current associated to global phase rotations:

i | ◁ Global phase rotations:

Eq. (7.90) is clearly invariant under global phase rotations of the spinors:

$$\Psi'(x) = e^{i\alpha}\Psi(x) \quad \text{for} \quad \alpha \in [0, 2\pi) \tag{7.95}$$

with infinitesimal generator $|\alpha| = |w| \ll 1$

$$\Psi'(x) = \Psi(x) + i\,w\Psi(x) \equiv \Psi(x) + w\,\delta\Psi(x) \quad \Rightarrow \quad \delta\Psi = i\,\Psi \tag{7.96}$$

→ Continuous symmetry:

$$\mathcal{L}_{\text{Dirac}}(\Psi, \partial\Psi) = \mathcal{L}_{\text{Dirac}}(\Psi', \partial\Psi') \tag{7.97}$$

ii | Noether theorem 6.85 → Conserved current density:

A straightforward calculation yields:

$$j^{\mu}_{\text{Dirac}} \overset{6.84}{=} -\frac{\partial\mathcal{L}_{\text{Dirac}}}{\partial(\partial_\mu\Psi_a)}\delta\Psi_a \overset{7.91}{=} \bar{\Psi}_b\gamma^{\mu}_{ba}\Psi_a = \bar{\Psi}\gamma^{\mu}\Psi\,. \tag{7.98}$$

$$\boxed{j^{\mu}_{\text{Dirac}} = \bar{\Psi}\gamma^{\mu}\Psi \quad \text{with} \quad \partial_\mu j^{\mu}_{\text{Dirac}} = 0} \tag{7.99}$$

Since the Lagrangian density $\mathcal{L}_{\text{Dirac}}$ is a Lorentz scalar, this Noether current must be a 4-vector. We can check this explicitly:

$$j'^{\mu}_{\text{Dirac}} = \bar{\Psi}'\gamma^{\mu}\Psi' = \bar{\Psi}\Lambda^{-1}_{\frac{1}{2}}\gamma^{\mu}\Lambda_{\frac{1}{2}}\Psi \overset{7.75}{=} \Lambda^{\mu}{}_{\nu}\bar{\Psi}\gamma^{\nu}\Psi = \Lambda^{\mu}{}_{\nu}j^{\mu}_{\text{Dirac}}\,. \tag{7.100}$$

iii | Conserved Noether charge:

$$Q = \int \mathrm{d}^3x\,\bar{\Psi}\gamma^0\Psi = \int \mathrm{d}^3x\,\underbrace{\Psi^\dagger\Psi}_{\geq 0} \geq 0 \tag{7.101}$$

→

$$\boxed{\text{Conserved norm on } L^2 \otimes \mathbb{C}^4\text{:} \quad \|\Psi\|^2 := \int \mathrm{d}^3x\,\Psi^\dagger\Psi} \tag{7.102}$$

- ¡! The positive-definite density $\Psi^\dagger\Psi = \bar{\Psi}\gamma^0\Psi$ is the time-component of a 4-vector and therefore *not* Lorentz invariant. However, the Noether charge Q is a Lorentz scalar so that the norm is Lorentz invariant: $\|\Psi'\| \overset{*}{=} \|\Psi\|$.

  Note that not all Noether charges are Lorentz scalars. The total field momentum Eq. (6.92), for example, is a 4-vector; similarly, the total field angular momentum Eq. (6.117) is a tensor of rank 2. However, it can be shown that the Noether charges of *internal* symmetries (like the $U(1)$ symmetry considered here) are necessarily Lorentz scalars (↑ *Coleman-Mandula theorem* [90]).

  Let us prove $Q'_a = Q_a$ in the case where the Noether current j^{μ}_a has no other Lorentz index (and the internal group generators commute with the generators of Lorentz transformations):

a | We consider an infinitesimal Lorentz transformation.

Coordinates transform according to Eq. (6.78),

$$\delta^{\alpha\beta} x^\mu = \frac{1}{2}\left(\eta^{\alpha\mu} x^\beta - \eta^{\beta\mu} x^\alpha\right) , \tag{7.103}$$

and, as a 4-vector, the *components* of the current transform in the same way:

$$\delta^{\alpha\beta} j_a^\mu = \frac{1}{2}\left(\eta^{\alpha\mu} j_a^\beta - \eta^{\beta\mu} j_a^\alpha\right) \mathrel{\hat{=}} j_a^\nu \left(\partial_\nu \delta^{\alpha\beta} x^\mu\right) . \tag{7.104}$$

(The labels a of the internal symmetry do not mix under this transformation because the internal symmetry is assumed to commute with Lorentz transformations.)

The generator of Lorentz transformations acts then according to Eq. (6.81) on the current *field*

$$-i G^{\alpha\beta} j_a^\mu(x) = \delta^{\alpha\beta} j_a^\mu - (\partial_\nu j_a^\mu)\delta^{\alpha\beta} x^\nu . \tag{7.105}$$

In the following we suppress the indices $\alpha\beta$ whenever possible.

b | It is easy to check that $\partial_\nu \delta x^\nu = 0$; furthermore, we know that $\partial_\nu j_a^\nu = 0$ from the Noether theorem. Together, this allows us to write the action of infinitesimal Lorentz transformations on the current as a 4-divergence:

$$-i G j_a^\mu(x) = \underbrace{(\partial_\nu j_a^\nu)}_{=0} \delta x^\mu + j_a^\nu (\partial_\nu \delta x^\mu) - (\partial_\nu j_a^\mu)\delta x^\nu - j_a^\mu \underbrace{(\partial_\nu \delta x^\nu)}_{=0} \tag{7.106a}$$

$$= \partial_\nu \left(j_a^\nu \delta x^\mu - j_a^\mu \delta x^\nu\right) . \tag{7.106b}$$

Here we used that $\delta j_a^\mu = j_a^\nu (\partial_\nu \delta x^\mu)$.

c | We finally obtain for the infinitesimal Lorentz transformation of the Noether charge:

$$-i G Q_a = \int \mathrm{d}^3x \, (-i G j_a^0) \tag{7.107a}$$

$$= \int \mathrm{d}^3x \, \partial_\nu \left(j_a^\nu \delta x^0 - j_a^0 \delta x^\nu\right) \tag{7.107b}$$

$$= \int \mathrm{d}^3x \, \partial_i \left(j_a^i \delta x^0 - j_a^0 \delta x^i\right) \tag{7.107c}$$

Gauss's theorem

$$= \int_\partial \mathrm{d}\sigma_i \left(j_a^i \delta x^0 - j_a^0 \delta x^i\right) = 0 \tag{7.107d}$$

In the last step we used that on the surface ∂ (typically spatial infinity) all fields vanish (for wavefunctions in L^2 this is clearly true).

Thus any Noether charge derived from internal symmetries transforms as a Lorentz scalar. In particular, the Dirac norm $\|\Psi\|$ is invariant under Lorentz transformations of the bispinor fields $\Psi(x)$.

13 | <u>Hamiltonian:</u>

i | Since the Dirac equation is first order in time, we can easily bring it into Schrödinger form and identify the Hamiltonian as the generator of time translations:

$$\text{Eq. (7.83)} \quad \Leftrightarrow \quad \left[i\hbar\gamma^0\partial_t + i\hbar c\gamma^i\partial_i - mc^2\right]\Psi = 0 \tag{7.108}$$

Use $(\gamma^0)^2 = \mathbb{1} \rightarrow$

$$i\hbar\partial_t \Psi = \left[-i\hbar c\gamma^0\gamma^i\partial_i + \gamma^0 mc^2\right]\Psi \tag{7.109}$$

ii | Let us define the new matrices:

$$\boxed{\beta := \gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \alpha_i := \gamma^0\gamma^i = \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \quad i = 1,2,3} \tag{7.110}$$

with $\beta^2 = \mathbb{1} = \alpha_i^2$ and $\{\alpha_i, \alpha_j\} = 0 = \{\alpha_i, \beta\}$ for $i \neq j$, and in particular

$$\beta^\dagger = \beta \quad \text{and} \quad \alpha_i^\dagger = \alpha_i \,. \tag{7.111}$$

¡! Note that the spatial gamma matrices are *anti-Hermitian*: $(\gamma^i)^\dagger = -\gamma^i$.

iii | With these matrices we can define the …

$$\boxed{\begin{aligned} &\text{⁂ } \textit{Dirac Hamiltonian}: \\ &\hat{H}_{\text{Dirac}} = -i\hbar c\, \vec{\alpha} \cdot \nabla + \beta\, mc^2 = c\, \vec{\alpha} \cdot \vec{p} + \beta\, mc^2 \end{aligned}} \tag{7.112}$$

with $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ and the ↓ *momentum operator* $\vec{p} = -i\hbar\nabla$.

→ The Dirac Hamiltonian is <u>Hermitian</u>:
(With respect to the standard inner product on $L^2 \otimes \mathbb{C}^4$):

$$\hat{H}^\dagger_{\text{Dirac}} = c\, \vec{\alpha}^\dagger \cdot \vec{p}^\dagger + \beta^\dagger\, mc^2 = c\, \vec{\alpha} \cdot \vec{p} + \beta\, mc^2 = \hat{H}_{\text{Dirac}} \tag{7.113}$$

Here we use that the momentum operator is self-adjoint (Hermitian) for (a dense subset of) functions in $L^2(\mathbb{R}^3, \mathbb{C})$:

$$\langle \psi | \vec{p}\phi \rangle = \int \mathrm{d}^3x\, \psi^*(-i\hbar\nabla\phi) = \int \mathrm{d}^3x\, (-i\hbar\nabla\psi)^*\phi = \langle \vec{p}\psi | \phi \rangle \,. \tag{7.114}$$

We used partial integration and $\lim\limits_{|\vec{x}|\to\infty} \phi(\vec{x}) = 0 = \lim\limits_{|\vec{x}|\to\infty} \psi(\vec{x})$ for admissible functions.

iv | The Dirac equation then takes the Schrödinger form

$$\boxed{i\hbar\partial_t \Psi(x) = \hat{H}_{\text{Dirac}} \Psi(x)} \tag{7.115}$$

In this form its Lorentz covariance is no longer manifest.

v | Eq. (7.102) conserved → ⋖ Inner product on $L^2 \otimes \mathbb{C}^4$:

$$\boxed{\langle \Psi | \Phi \rangle := \int \mathrm{d}^3x\, \Psi^\dagger(t,\vec{x})\Phi(t,\vec{x}) \quad \text{with} \quad \|\Psi\| = \sqrt{\langle \Psi | \Psi \rangle}} \tag{7.116}$$

This inner product is constant under the evolution of the Dirac equation:

$$\text{Eq. (7.113) \& Eq. (7.115)} \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t} \langle \Psi | \Phi \rangle \mathrel{\hat{=}} 0 \tag{7.117}$$

- This generalizes our previous finding in Eq. (7.102) about the conserved norm.

- That the inner product is constant is straightforward to show:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\Psi|\Phi\rangle = \int \mathrm{d}^3x \left[\Psi^\dagger\dot{\Phi} + \dot{\Psi}^\dagger\Phi\right] \tag{7.118a}$$

$$\overset{7.115}{=} \frac{1}{i\hbar}\int \mathrm{d}^3x \left[\Psi^\dagger\left(\hat{H}_{\mathrm{Dirac}}\Phi\right) - \left(\hat{H}_{\mathrm{Dirac}}\Psi\right)^\dagger\Phi\right] \tag{7.118b}$$

$$\overset{7.113}{=} \frac{1}{i\hbar}\int \mathrm{d}^3x \left[\Psi^\dagger\left(\hat{H}_{\mathrm{Dirac}}\Phi\right) - \Psi^\dagger\left(\hat{H}_{\mathrm{Dirac}}\Phi\right)\right] = 0 \tag{7.118c}$$

14 | Conclusion:

Let us summarize our findings and compare them to the Klein-Gordon equation:

	Klein-Gordon Equation $(\partial^2 + M^2)\phi = 0$	**Dirac Equation** $(i\not{\partial} - M)\Psi = 0$
Time derivative	second order	first order
Function space	$L^2(\mathbb{R}^{1,3}, \mathbb{C})$	$L^2(\mathbb{R}^{1,3}, \mathbb{C}^2 \oplus \mathbb{C}^2)$
Wavefunction	Complex scalar field $\phi(x)$	Complex bispinor field $\Psi(x)$
Conserved form	$i\int \mathrm{d}^3x\ (\phi_1^*\dot{\phi}_2 - \dot{\phi}_1^*\phi_2)$	$\int \mathrm{d}^3x\ \Psi_1^\dagger\Psi_2$
Positive definite?	✗	✓
Hermitian Hamiltonian?	✗	✓

→ What about the *eigenenergies* and *eigenstates* of $\hat{H}_{\mathrm{Dirac}}$?

7.2.1. Free-particle solutions of the Dirac equation

15 | Eq. (7.86): Solutions of the Dirac equation satisfy the Klein-Gordon equation component-wise:

$\xrightarrow{\text{Eq. (7.34)}}$ Ansatz:

$$\Psi^\pm(x) = \psi^\pm(p)e^{\mp\frac{i}{\hbar}px} \quad \text{with} \quad p^0 = \frac{E}{c} = \sqrt{\vec{p}^2 + m^2c^2} > 0 \tag{7.119}$$

with complex-valued four-component ⁂ *bispinor*

$$\psi^\pm(p) \equiv \begin{pmatrix}\psi_L^\pm \\ \psi_R^\pm\end{pmatrix} \in \mathbb{C}^4 \simeq \mathbb{C}^2 \oplus \mathbb{C}^2\,. \tag{7.120}$$

- We set $p^0 > 0$ for both positive $(+)$ and negative $(-)$ energy/frequency solutions and change the sign of p in the exponent (to simplify the discussion below).
- Note that $px = p_\mu x^\mu = Et - \vec{p}\cdot\vec{x}$.

16 | Eq. (7.119) in Eq. (7.83) yields:

$$(\pm\gamma^\mu p_\mu - mc)\psi^\pm(p) = \begin{pmatrix}-mc & \pm p\sigma \\ \pm p\bar{\sigma} & -mc\end{pmatrix}\begin{pmatrix}\psi_L^\pm \\ \psi_R^\pm\end{pmatrix} = 0 \tag{7.121}$$

with $p\sigma = p_\mu\sigma^\mu$ and $\sigma^\mu = (\mathbb{1}, \sigma^x, \sigma^y, \sigma^z)$ and $\bar{\sigma}^\mu = (\mathbb{1}, -\sigma^x, -\sigma^y, -\sigma^z)$.

17 | Mathematical facts (check these!):

- $(p\sigma)(p\bar{\sigma}) \mathrel{\hat{=}} p^2 = m^2c^2$
- Eigenvalues of $p\sigma$ and $p\bar{\sigma}$: $p^0 \pm |\vec{p}| \to$ for $p^0 > 0$ and $m \neq 0$ *positive spectrum*

  $\to$ $p\sigma$ and $p\bar{\sigma}$ are *invertible* and the positive square roots $\sqrt{p\sigma}$ and $\sqrt{p\bar{\sigma}}$ are Hermitian.

18 | $\triangleleft$ $\psi_L^\pm \equiv \sqrt{p\sigma}\,\xi^\pm$ with arbitrary, normalized $[(\xi^\pm)^\dagger\xi^\pm = 1]$ ⁂ *spinor* $\xi^\pm \in \mathbb{C}^2$:

$$\text{Eq. (7.121)} \quad \Rightarrow \quad -mc\sqrt{p\sigma}\,\xi^\pm \pm p\sigma\,\psi_R^\pm = 0 \tag{7.122}$$

Use $\sqrt{p\sigma}\sqrt{p\bar{\sigma}} = mc$:

$$\psi_R^\pm = \pm\frac{mc}{\sqrt{p\sigma}}\xi^\pm = \pm\sqrt{p\bar{\sigma}}\xi^\pm \tag{7.123}$$

$\to$ $\psi_L^\pm$ and $\psi_R^\pm$ are now parametrized by the spinor $\xi^\pm \in \mathbb{C}^2$ (which is unconstrained!).

The second equation in Eq. (7.121) yields the same solution.

19 | <u>Solutions:</u>

Let us adopt the more conventional notation

$$\begin{matrix}\xi^+ \mapsto \xi \\ \xi^- \mapsto \eta\end{matrix} \quad \text{and} \quad \begin{matrix}\psi^+ \mapsto u \\ \psi^- \mapsto v\end{matrix} \tag{7.124}$$

and choose the spinor basis ξ^s, η^s $(s = \uparrow, \downarrow)$ with

$$\xi^\uparrow, \eta^\uparrow = \begin{pmatrix}1\\0\end{pmatrix} \quad \text{and} \quad \xi^\downarrow, \eta^\downarrow = \begin{pmatrix}0\\1\end{pmatrix}. \tag{7.125}$$

Then linearly independent solutions of the free Dirac equation can be written as:

$$\Psi^+_{\vec{p},s}(x) = \underbrace{\begin{pmatrix}\sqrt{p\sigma}\xi^s \\ \sqrt{p\bar{\sigma}}\xi^s\end{pmatrix}}_{u^s(\vec{p})} e^{-\frac{i}{\hbar}px} \qquad \text{(positive energy solutions)} \tag{7.126a}$$

$$\Psi^-_{\vec{p},s}(x) = \underbrace{\begin{pmatrix}\sqrt{p\sigma}\eta^s \\ -\sqrt{p\bar{\sigma}}\eta^s\end{pmatrix}}_{v^s(\vec{p})} e^{+\frac{i}{\hbar}px} \qquad \text{(negative energy solutions)} \tag{7.126b}$$

with $p^\mu = (p^0, \vec{p})$, $p^0 = \sqrt{\vec{p}^2 + m^2c^2} > 0$ and $s = \uparrow, \downarrow$.

$\to$ *Four* linearly independent solutions for each 3-momentum $\vec{p}$ ($\pm$ and $s = 1, 2$).

You can easily check that Eq. (7.126) form an orthogonal eigenbasis of the Dirac Hamiltonian:

$$\hat{H}_{\text{Dirac}}\Psi^\pm_{\vec{p},s} \mathrel{\hat{=}} \pm E_{\vec{p}}\Psi^\pm_{\vec{p},s} \quad \text{with spectrum} \quad E_{\vec{p}} = \sqrt{p^2c^2 + m^2c^4}\,. \tag{7.127}$$

Their orthogonality follows with the identities

$$[u^r(\vec{p})]^\dagger u^s(\vec{p}) \mathrel{\hat{=}} \tfrac{2}{c}E_{\vec{p}}\delta^{rs}\,, \quad [v^r(\vec{p})]^\dagger v^s(\vec{p}) \mathrel{\hat{=}} \tfrac{2}{c}E_{\vec{p}}\delta^{rs}\,, \quad [u^r(\vec{p})]^\dagger v^s(-\vec{p}) \mathrel{\hat{=}} 0\,. \tag{7.128}$$

$$\boxed{\to \text{The Dirac equation still has } \textit{negative-energy solutions}.} \tag{7.129}$$

20 | Interpretation:

- The negative energy solutions are not problematic as long as we consider a single particle (electron) without interactions (this is also why we can apply the Dirac equation to describe the hydrogen atom, → *below*). However, in reality the electron couples to a dynamic electromagnetic field and therefore could emit a photon (thereby lowering its energy). If the negative energy eigenstates really exist, there is no reason why this process should terminate; as a consequence, no stable electrons should exist.

  Dirac writes in Ref. [91]:

  > *It is true that in the case of a steady electromagnetic field we can draw a distinction between those solutions [..] with E positive and those with E negative and may assert that only the former have a physical meaning (as was actually done when the theory was applied to the determination of the energy levels of the hydrogen atom), but if a perturbation is applied to the system it may cause transitions from one kind of state to the other. In the general case of an arbitrarily varying electromagnetic field we can make no hard-and-fast separation of the solutions of the wave equation into those referring to positive and those to negative kinetic energy. Further, in the accurate quantum theory in which the electromagnetic field also is subjected to quantum laws, transitions can take place in which the energy of the electron changes from a positive to a negative value even in the absence of any external field, the surplus energy [..] being spontaneously emitted in the form of radiation. [..] Thus we cannot ignore the negative-energy states without giving rise to ambiguity in the interpretation of the theory.*

  Dirac suggested a "fix" for this problem [91]: Because the electron is a fermion, it obeys the Pauli exclusion principle. Thus one can imagine that (for some reason) all the negative energy states *are already occupied* by electrons. The electrons we see can then only occupy the positive energy states and cannot decay to states of arbitrarily low energy. This construct is know as the ↑ *hole theory* because creating a "hole" in this ↑ *Dirac sea* of electrons with negative energy can be viewed as an excitation with *positive* energy. Dirac's holes are of course a precursor to what we know today as ↑ *antiparticles*. (Dirac didn't think of it this way, he conjectured that the holes in his sea of electrons are the *protons*!)

- However, Dirac's interpretation is not how we deal with the negative-energy solutions today: Within the modern framework of ↑ *relativistic quantum field theories*, the four single-particle wave functions are associated (through "second" quantization of the Dirac field and the construction of a fermionic ↑ *Fock space*) to two particle types, both with *positive* energy and two internal spin-$\frac{1}{2}$ states:

	Type	Momentum	Spin	Energy	Charge
$\Psi^+_{\vec{p},\uparrow}$:	fermion	$+\vec{p}$	$+\frac{1}{2}$	$+E_{\vec{p}}$	$+1$
$\Psi^+_{\vec{p},\downarrow}$:	fermion	$+\vec{p}$	$-\frac{1}{2}$	$+E_{\vec{p}}$	$+1$
$\Psi^-_{\vec{p},\uparrow}$:	antifermion	$+\vec{p}$	$-\frac{1}{2}$	$+E_{\vec{p}}$	-1
$\Psi^-_{\vec{p},\downarrow}$:	antifermion	$+\vec{p}$	$+\frac{1}{2}$	$+E_{\vec{p}}$	-1

(7.130)

  Here "Spin" refers to the ↓ *spin-polarization quantum number* $m_z = \pm\frac{1}{2}$.

  → *Take home message:*

  Relativistic quantum mechanics predicts *spin* and *antiparticles*. (7.131)

The negative energy solutions (and therefore the existence of antiparticles) are a necessary feature of relativistic quantum mechanics (more precisely: relativistic quantum field theories, via the ↑ *CPT-theorem*).

By contrast, the fact that particles can have an internal angular momentum (spin), and that this angular momentum can take half-integer values $S = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ is not a relativistic feature *per se*: Spin enters quantum mechanics the moment one considers *spatial rotations* and its representations on the Hilbert space. Because these can be ↑ *projective*, one is forced to study the irreducible linear representations of SU(2) – the double cover of the rotation group SO(3) – which happen to be labeled by the "spin quantum numbers" $S = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. Now, since the rotation group is a subgroup of the homogeneous Lorentz group, $\mathrm{SO}(3) \subset \mathrm{SO}^+(1,3)$, the moment a quantum theory is relativistic [i.e., features a representation of $\mathrm{SO}^+(1,3)$], spin enters the stage automatically. However, you can describe quantum particles with spin *without* making quantum mechanics relativistic.

- The Dirac equation applies to all spin-$\frac{1}{2}$ fermions. The most prominent example is of course the electron e^- and its associated antiparticle, the positron e^+. However, all other elementary fermions, namely leptons (like the muon/antimuon, the tau/antitau and the neutrinos) and the six quark/antiquark pairs, are described by the Dirac equation as well.

7.2.2. The relativistic hydrogen atom

21 | Dirac equation with a static EM field:

To couple the Dirac field Ψ in a gauge- and covariant way to a static EM field A^μ, we use the same trick as for the Klein-Gordon equation:

← *Minimal coupling* Eq. (7.38) →

$$\partial_\mu \mapsto D_\mu = \partial_\mu + iQA_\mu \quad \Rightarrow \quad \not{\partial} \mapsto \not{D} = \not{\partial} + iQ\not{A} = \gamma^\mu \partial_\mu + iQ\gamma^\mu A_\mu \tag{7.132}$$

For an electron it is $Q = -\frac{e}{\hbar c}$ with $e > 0$.

→

$$\boxed{(i\not{D} - M)\Psi = 0} \tag{7.133}$$

In this form, the Dirac equation is manifest Lorentz- and gauge invariant.

We can expand Eq. (7.133) to obtain a less abstract (but more convoluted) expression:

$$\left[i\gamma^\mu \partial_\mu - Q\gamma^\mu A_\mu - M\right]\Psi = 0 \tag{7.134a}$$

$$\Leftrightarrow \quad \left[i\hbar\gamma^0\partial_t + i\hbar c\gamma^i\partial_i - q\gamma^0\varphi + q\gamma^i A_i - mc^2\right]\Psi = 0 \tag{7.134b}$$

$$\Leftrightarrow \quad \left[i\hbar\partial_t + i\hbar c\,\vec{\alpha}\cdot\nabla - q\,\varphi + q\,\vec{\alpha}\cdot\vec{A} - \beta\,mc^2\right]\Psi = 0 \tag{7.134c}$$

Here we used $Q = \frac{q}{\hbar c}$, $M = \frac{mc}{\hbar}$, and $A_\mu = (\varphi, -\vec{A})$; q is the charge of the particle.

In Schrödinger form the Dirac equation reads then:

$$i\hbar\partial_t\Psi = \left[-i\hbar c\,\vec{\alpha}\cdot\nabla + q\,\varphi - q\,\vec{\alpha}\cdot\vec{A} + \beta\,mc^2\right]\Psi \tag{7.135a}$$

$$i\hbar\partial_t\Psi = \underbrace{\left[c\,\vec{\alpha}\cdot\left(\vec{p} - \frac{q}{c}\vec{A}\right) + q\,\varphi + \beta\,mc^2\right]}_{\hat{H}_{\mathrm{Dirac},A}}\Psi \tag{7.135b}$$

22 | Choose the Coulomb potential (of the proton)

$$\varphi(x) = \frac{e}{|\vec{x}|} \quad \text{and} \quad \vec{A} = \vec{0} \tag{7.136}$$

and set $q = -e$ (charge of the electron) →

$$i\hbar\partial_t \Psi = \left[-i\hbar c\,\vec{\alpha}\cdot\nabla - \frac{e^2}{|\vec{x}|} + \beta\, mc^2\right]\Psi \tag{7.137}$$

With the ansatz $\Psi(t,\vec{x}) = \psi(\vec{x})e^{-\frac{i}{\hbar}Et}$ one obtains the time-independent eigenvalue problem

$$\left[-i\hbar c\,\vec{\alpha}\cdot\nabla - \frac{e^2}{|\vec{x}|} + \beta\, mc^2 - E\right]\psi(\vec{x}) = 0 \quad \text{with} \quad \psi = \begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix} : \mathbb{R}^3 \to \mathbb{C}^4 . \tag{7.138}$$

Note that β (unlike α_i) is an off-diagonal block matrix that mixes the two spinors ψ_+ and ψ_-; this complicates the solution. However, one can solve Eq. (7.138) *exactly* and compute the eigenvalues E and eigenstates $\psi(\vec{x})$.

23 | Solution: $\xrightarrow{*}$ Eigenenergies (including the rest energy of the electron):

$$\boxed{E_{n,j} = mc^2\left\{1 + \frac{\alpha^2}{\left[n - j - \frac{1}{2} + \sqrt{\left(j+\frac{1}{2}\right)^2 - \alpha^2}\right]^2}\right\}^{-\frac{1}{2}}} \tag{7.139}$$

with

- ↓ *principal quantum number* $n = 1, 2, \ldots$
- ↓ *total angular momentum quantum number* $j = \frac{1}{2}, \frac{3}{2}, \ldots, n - \frac{1}{2}$
- ↓ *fine-structure constant* $\alpha \approx \frac{1}{137}$

The principal quantum number $n = 1, 2, \ldots$ constrains the allowed *orbital* angular momentum to $l = 0, 1, \ldots, n-1$. The allowed *total* angular momentum is then given by the usual rules of angular momentum addition: $|l - \frac{1}{2}| \leq j \leq |l + \frac{1}{2}|$ (in integer steps, $s = \frac{1}{2}$ is the electron spin).

So for example $n = 1$ allows only for $l = 0$ and therefore $j = \frac{1}{2}$; this is the $1S_{1/2}$ orbital and the ground state of the hydrogen atom. For $n = 2$ one finds again $l = 0$ with $j = \frac{1}{2}$ (the $2S_{1/2}$ orbital) but also $l = 1$ with $j = \frac{1}{2}$ and $j = \frac{3}{2}$ (the $2P_{1/2}$ and $2P_{3/2}$ orbitals – which are no longer degenerate because $E_{2,1/2} \neq E_{2,3/2}$).

This result explains why in the hydrogen spectrum the degeneracy of the $2S_{1/2}$ and $2P_{3/2}$ orbitals is lifted whereas the $2S_{1/2}$ orbital remains degenerate with the $2P_{1/2}$ orbital (← *fine-structure*).

→

$$\boxed{\text{The Dirac equation explains the fine-structure of the hydrogen atom ☺.}} \tag{7.140}$$

Note: You may have encountered the following Hamiltonian for the hydrogen atom with added *relativistic corrections*:

$$\hat{H}_{\text{rel}} = \underbrace{\frac{\vec{p}^2}{2m} - \frac{e^2}{r}}_{\substack{\text{Non-rel.}\\ \text{hydrogen atom}}} \underbrace{\underbrace{-\frac{1}{2mc^2}\left(\frac{p^2}{2m}\right)^2}_{\text{Rel. kinetic energy}} + \underbrace{\frac{e^2}{2m^2c^2}\frac{\vec{L}\cdot\vec{S}}{r^3}}_{\text{Spin-orbit coupling}} - \underbrace{\frac{e^2\hbar^2}{8m^2c^2}\Delta\left(\frac{1}{r}\right)}_{\text{Darwin term}}}_{\text{Relativistic corrections}} . \tag{7.141}$$

This Hamiltonian can reproduce the fine-structure as well. It has several drawbacks, though:

- It is only an *approximation*.
- It is hard to solve (perturbation theory!).
- The Schrödinger equation $i\hbar\partial_t\psi = \hat{H}_{\text{rel}}\psi$ is not manifestly Lorentz covariant.
- The relativistic corrections are *ad hoc* and seemingly independent of each other.

Luckily, Eq. (7.141) does not have to appear out of thin air; one can show via a complicated derivation (↑ *Foldy-Wouthuysen transformation*) that it is indeed the non-relativistic limit [with corrections in order $(v/c)^2$] of the Dirac equation Eq. (7.138) in a Coulomb potential (without the rest energy mc^2 of the electron).

7.2.3. The electron g-factor

Besides the fine structure, there is one other "mystery" that the relativistic treatment of the electron in terms of the Dirac equation finally explains: The non-classical ratio between the electrons internal magnetic moment and its spin.

24 | ⊲ Dirac electron in homogeneous magnetic field $\vec{B} = \nabla \times \vec{A}$ ($\varphi = 0$):

$$\text{Eq. (7.135b)} \quad \xRightarrow{\Psi = \psi e^{-\frac{i}{\hbar}Et}} \quad \left[c\,\vec{\alpha}\cdot\left(\vec{p} + \tfrac{e}{c}\vec{A}\right) + \beta\, mc^2 - E\right]\psi = 0 \tag{7.142}$$

with bispinor

$$\psi = \begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix} : \mathbb{R}^3 \to \mathbb{C}^4\,. \tag{7.143}$$

Using Eq. (7.110) we can write this equation in terms of the two spinors:

$$\left(-c\vec{\sigma}\vec{\pi} - E\right)\psi_L + mc^2\psi_R = 0 \tag{7.144a}$$
$$\left(+c\vec{\sigma}\vec{\pi} - E\right)\psi_R + mc^2\psi_L = 0 \tag{7.144b}$$

Here we used $\vec{\pi} = \vec{p} + \frac{e}{c}\vec{A}$ and introduced $\vec{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$.

We can now use one of the two equations to decouple the system:

$$\left(c\vec{\sigma}\vec{\pi} + E\right)\left(c\vec{\sigma}\vec{\pi} - E\right)\psi_R + (mc^2)^2\psi_R = 0 \tag{7.145a}$$
$$\Leftrightarrow \quad c^2(\vec{\sigma}\vec{\pi})^2\psi_R - \left[E^2 - (mc^2)^2\right]\psi_R = 0 \tag{7.145b}$$

25 | $\overset{\circ}{\to}$ Non-relativistic approximation:

We can use $E^2 - (mc^2)^2 = (E - mc^2)(E + mc^2) \approx 2mc^2\tilde{E}$ with $\tilde{E} = E - mc^2$ to find a non-relativistic approximation of Eq. (7.145b):

$$\frac{1}{2m}(\vec{\sigma}\vec{\pi})^2\psi_R = \tilde{E}\psi_R \tag{7.146}$$

Last, use the Pauli algebra $\sigma^i\sigma^j = \delta^{ij} + i\varepsilon^{ijk}\sigma^k$ and $B_k = \varepsilon_{ijk}(\partial_i A_j)$ to show that $(\vec{\sigma}\vec{\pi})^2 \mathrel{\hat{=}} \vec{\pi}^2 + \frac{\hbar e}{c}\vec{\sigma}\cdot\vec{B}$. We end up with the non-relativistic, time-independent Schrödinger equation of a charged particle in a magnetic field with a spin-dependent Zeeman term:

$$\Bigg[\underbrace{\frac{1}{2m}\left(\vec{p} + \frac{e}{c}\vec{A}\right)^2}_{\text{Particle in mag. field}} + \underbrace{\frac{e\hbar}{2mc}\vec{\sigma}\cdot\vec{B}}_{\text{Zeeman effect}}\Bigg]\psi_R = \tilde{E}\psi_R \tag{7.147}$$

→ Potential energy of electron in magnetic field:

$$E_{\text{mag}} \overset{\text{def}}{=} -\vec{\mu}\cdot\vec{B} \overset{7.147}{=} \frac{e\hbar}{2mc}\vec{\sigma}\cdot\vec{B} \tag{7.148}$$

26 | → Magnetic moment (operator) of the electron:

$$\vec{\mu}_e = -\frac{e\hbar}{2mc}\vec{\sigma} = g_e\frac{\mu_B}{\hbar}\vec{S} \tag{7.149}$$

with ↓ *spin operator* $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$ and ↓ *Bohr magneton* $\mu_B = \frac{e\hbar}{2mc}$ and

$$\boxed{\text{⁂ } \textit{Electron } g\textit{-factor} \quad g_e = -2\,.} \tag{7.150}$$

27 | Comments:

- What makes Eq. (7.149) with $g_e = -2$ remarkable is that it is *not* what one would expect if the magnetic moment would be caused by a charge flying along a tiny orbit with angular momentum $\vec{S}$. Indeed, a straightforward classical calculation yields for the relation between magnetic moment and (orbital) angular momentum $\vec{L}$:

$$\vec{\mu}_L = g_L\frac{\mu_B}{\hbar}\vec{L} \quad \text{with} \quad g_L = -1\,. \tag{7.151}$$

  So, quite surprisingly, the Dirac equation predicts that the internal angular momentum (= spin) produces "twice as much" magnetic moment as one would naïvely expect.

  That this really is the case can be easily measured: Just apply a magnetic field to hydrogen atoms and observe how strongly their spectral lines split as a function of the magnetic field strength (↑ *anomalous Zeeman effect*). This effect had already been experimentally observed at the end of the 19th century [92, 93]. Since it was unknown at the time that electrons had spin, certain line splittings could not be explained (therefore "anomalous"). The fact that the Dirac equation explains both – the electron spin and its "non-classical" g-factor – is therefore a remarkable feature of relativistic quantum mechanics.

- If one measures the electron g-factor *really, really* precisely, one finds [94]

$$g_e = -2.00231930436118(27)\,. \tag{7.152}$$

  You may notice that this is not *exactly* -2 but a tiny bit off. One cannot explain this deviation with the Dirac equation because it stems from "virtual particles" that modify how the electron interacts with the EM field (and the Dirac equation is a single-particle wave equation). It is therefore remarkable that modern theories *can* explain this deviation perfectly (up to error bars), but for this one needs the machinery of ↑ *relativistic quantum field theory*.

Part II.

General Relativity