↓Lecture 12 [16.01.24]

6. Relativistic Field Theories I: Electrodynamics

6.1. A primer on classical field theories

We start with a general discussion of classical field theories on Minkowski space; Maxwell's electrodynamics is the prime example for such theories (→ *next section*).

Details: Chapter 1 of my QFT script [19]

6.1.1. Remember: Classical mechanics of "points"

With "points" we mean a discrete set of degrees of freedom.

1 | ⊲ Degrees of freedom q_i labeled by $i = 1, \dots, N$

2 | Lagrangian $L(\{q_i\}, \{\dot{q}_i\}, t) = T - V$
We write q for $\{q_i\} = \{q_1, \dots, q_N\}$. T is the kinetic, V the potential energy.

3 | Action $S[q] = \int \mathrm{d}t\, L(q(t), \dot{q}(t), t) \in \mathbb{R}$
This is a *functional* of trajectories $q = q(t)$.

4 | Hamilton's principle of least action:

$$\frac{\delta S[q]}{\delta q} \overset{!}{=} 0 \quad \Leftrightarrow \quad \delta S = \int \mathrm{d}t\, \delta L \overset{!}{=} 0 \tag{6.1}$$

δ denotes functional derivatives/variations.

5 | Euler-Lagrange equations ($i = 1, \dots, N$):

$$\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_i} = 0 \tag{6.2}$$

6.1.2. Analogous: Lagrangian Field Theory

Now we consider a *continuous* set of degrees of freedom:

6 | ⊲ One or more fields $\phi(x)$ on spacetime $x \in \mathbb{R}^{1,3}$ with derivatives $\partial_\mu \phi(x)$
If there are multiple fields we label them by indices: $\phi_k(x)$.
In the following we suppress these indices for the sake of simplicity.

7 | ⁂ *Lagrangian density* $\mathcal{L}(\phi, \partial\phi, x)$
Most general form: $\mathcal{L}(\{\phi_k\}, \{\partial_\mu \phi_k\}, \{x^\mu\})$. (No explicit x^μ-dependence in the following!)
→ Lagrangian $L = \int_{\text{Space}} \mathrm{d}^3x\, \mathcal{L}(\phi, \partial\phi)$
(We omit the "density" in the following.)

8 | Action:

$$S[\phi] = \int \mathrm{d}t\, L = \int \mathrm{d}t\, \mathrm{d}^3x\ \mathcal{L}(\phi, \partial\phi) = \frac{1}{c}\int_{\text{Spacetime}} \mathrm{d}^4x\ \mathcal{L}(\phi, \partial\phi) \tag{6.3}$$

$S[\phi]$ is a functional of "field trajectories" in $\mathbb{R}^{1,3}$.

9 | Action principle:

The classical field evolutions of the system extremize the action:

$$\delta S[\phi] \stackrel{!}{=} 0 \tag{6.4}$$

This variation can be evaluated along the same lines as for the classical mechanics of points:

$$0 \stackrel{!}{=} \delta S[\phi] = \int \mathrm{d}^4x\, \delta\mathcal{L} \tag{6.5a}$$

$$= \int \mathrm{d}^4x \left\{\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi)\right\} \tag{6.5b}$$

Add zero and use $\delta(\partial_\mu\phi) = \partial_\mu(\delta\phi)$

$$= \int \mathrm{d}^4x \left\{\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\delta\phi + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right)\right\} \tag{6.5c}$$

Gauss theorem

$$= \int_{\text{Boundary}} \mathrm{d}\sigma_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \underbrace{\delta\phi}_{=0} + \int \mathrm{d}^4x \underbrace{\left\{\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\right\}}_{=0}\delta\phi \tag{6.5d}$$

- Note that ϕ is fixed on the boundary and therefore $\delta\phi = 0$.
- The second term vanishes because the integral must vanish for arbitrary variations $\delta\phi$.

10 | Euler-Lagrange equations (one for each field):

$$\boxed{\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = 0} \tag{6.6}$$

- Note the *Einstein summation* over repeated indices.
- These equations are manifestly Lorentz covariant if $\mathcal{L}$ is a Lorentz scalar; such field theories are called ⁂ *relativistic field theories*.
- If there are multiple fields ϕ_k, there is one Euler-Lagrange equation per field (it is straightforward to generalize the derivation above).

11 | Hamiltonian formalism:

Just like for the mechanics of points, we can define:

$$\boxed{\pi := \frac{\partial\mathcal{L}}{\partial\dot{\phi}} \qquad ⁂\ \textit{Momentum density}\ \text{conjugate to}\ \phi} \tag{6.7}$$

Like $\phi(x)$, the momentum is a *field*: $\pi(x)$. Here it is $\dot{\phi}(x) \equiv \partial_0\phi(x)$.

→

$$\boxed{\mathcal{H}(\pi, \phi, \nabla\phi) := \pi\dot{\phi} - \mathcal{L}(\phi, \partial\phi) \qquad ⁂\ \textit{Hamiltonian density}} \tag{6.8}$$

- Here $\dot{\phi}$ is to be expressed as a function of the conjugate momentum via Eq. (6.7).
- The argument $\partial\phi$ of $\mathcal{L}$ is short for $\{\partial_\mu\phi\}$ or $\{\nabla\phi, \dot{\phi}\}$.

→

$$\boxed{H := \int \mathrm{d}^3x\, \mathcal{H} \qquad \text{⁂ } \textit{Hamiltonian}} \tag{6.9}$$

For given fields $\pi(x)$ and $\phi(x)$, H is a (potentially constant) function of time. By contrast, the Hamiltonian *density* $\mathcal{H}$ is a function of space $\vec{x}$ and time t.

6.2. Electrodynamics: Covariant formulation and Lagrange function

We now want to reformulate Maxwell's electrodyamics in this formalism, i.e., we want to find a Lagrangian density (and an associated action) such that the Euler-Lagrange equations are the Maxwell equations.

1 | Remember:

i | ↓ *Maxwell equations* (in cgs units):

$$\boxed{\begin{aligned} \text{Magnetic Gauss's law:} \quad & \nabla\cdot\vec{B} = 0 && (6.10a)\\ \text{Maxwell-Faraday law:} \quad & \nabla\times\vec{E} + \tfrac{1}{c}\partial_t\vec{B} = 0 && (6.10b)\\ \text{Electric Gauss's law:} \quad & \nabla\cdot\vec{E} = 4\pi\rho && (6.10c)\\ \text{Ampère's law:} \quad & \nabla\times\vec{B} - \tfrac{1}{c}\partial_t\vec{E} = \tfrac{4\pi}{c}\vec{j} && (6.10d)\end{aligned}}$$

with charge density $\rho(x)$ and current density $\vec{j}(x)$ that satisfy the ⁂ *continuity equation*

$$\boxed{\partial_t\rho + \nabla\cdot\vec{j} = 0\,.} \tag{6.11}$$

This follows from the two inhomogeneous Maxwell equations Eqs. (6.10c) and (6.10d). Note that here ρ and $\vec{j}$ are external fields and not dynamic degrees of freedom. The statement is therefore that only for external fields that satisfy Eq. (6.11) the Maxwell equations yield solutions for $\vec{E}$ and $\vec{B}$.

ii | Homogeneous Maxwell equations (HME) Eq. (6.10a) & Eq. (6.10b)

$\overset{\circ}{\to}$ ∃ "Scalar" potential φ and "Vector" potential $\vec{A}$:

$$\vec{E} = -\nabla\varphi - \tfrac{1}{c}\partial_t\vec{A} \quad \text{and} \quad \vec{B} = \nabla\times\vec{A} \tag{6.12}$$

- Constraining the fields $\vec{E}$ and $\vec{B}$ to this form satisfies the *homogeneous Maxwell equations* Eqs. (6.10a) and (6.10b) automatically.
- Because of the two homogeneous Maxwell equations, the *six* fields $\{E_x, E_y, E_z, B_x, B_y, B_z\}$ are not independent so that all degrees of freedom can be encoded in the *four* fields $\{\varphi, A_x, A_y, A_z\}$. This suggests a reformulation of Maxwell's theory in terms of these "potentials".

iii | Gauge transformation:

$\lhd$ Arbitrary function $\lambda : \mathbb{R}^{1,3} \to \mathbb{R}$ and define

$$\vec{A}' := \vec{A} + \nabla\lambda \quad \text{and} \quad \varphi' := \varphi - \tfrac{1}{c}\partial_t\lambda \tag{6.13}$$

This transformation of fields is called a ⁂ *gauge transformation* (→ *below*).

$\overset{\circ}{\to}$ $\vec{E} = \vec{E}'$ and $\vec{B} = \vec{B}'$

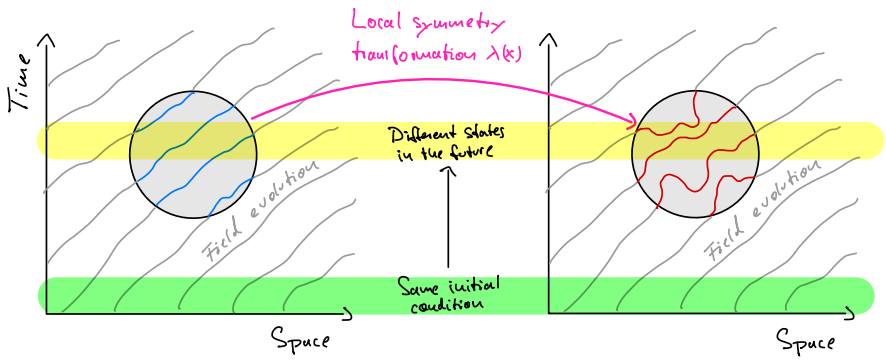
$\to$ The potentials φ and $\vec{A}$ are not unique.

iv | Inhomogeneous Maxwell equations (IME) Eqs. (6.10c) and (6.10d) in terms of the potentials:

$$\text{Eq. (6.10c)} \quad \Leftrightarrow \quad \nabla^2\varphi + \tfrac{1}{c}\partial_t(\nabla\cdot\vec{A}) = -4\pi\rho \tag{6.14a}$$

$$\text{Eq. (6.10d)} \quad \Leftrightarrow \quad \nabla^2\vec{A} - \tfrac{1}{c^2}\partial_t^2\vec{A} = -\tfrac{4\pi}{c}\vec{j} + \nabla\left(\nabla\cdot\vec{A} + \tfrac{1}{c}\partial_t\varphi\right) \tag{6.14b}$$

In this form, electrodynamics is a *gauge theory* because it has a *local* symmetry, namely the transformation Eq. (6.13). Indeed, it is straightforward to show that if $(\varphi, \vec{A})$ is a solution of Eq. (6.14), then $(\varphi', \vec{A}')$ given by Eq. (6.13) is another solution. Since $\lambda(x)$ is arbitrary, one can choose continuously differentiable $\lambda(x)$ that vanish everywhere except for a compact region of spacetime. This makes Eq. (6.13) a *local* symmetry transformation of the PDE system Eq. (6.14); such local symmetries are called ⁂ *gauge transformations*, and models that feature such symmetries are refered to as ⁂ *gauge theories*. The locality of the symmetry has profound implications:



Thus, if we want a deterministic theory (meaning: a theory with predictive power), we *cannot* interpret the gauge fields $(\varphi, \vec{A})$ as physical (= observable) degrees of freedom. Our only choice (to save predictability) is to identify the *equivalence classes* $[(\varphi, \vec{A})]$ of field configurations that are related by (local) gauge transformations as physical states; this is the defining property of a gauge theory. In a nutshell: local symmetries must be interpreted as gauge symmetries and fields related by such transformations are mathematically redundant descriptions of the *same* physical state.

v | Eq. (6.14) Gauge theory → Fix a gauge:

$$\boxed{\nabla\cdot\vec{A} + \tfrac{1}{c}\partial_t\varphi \overset{!}{=} 0 \quad ⁂\ \textit{Lorenz gauge (LG)}} \tag{6.15}$$

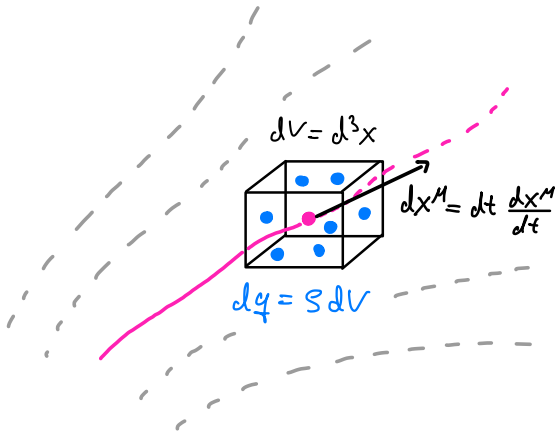
It is straightforward to show that for any given $(\varphi, \vec{A})$ there is a gauge transformation λ such that $(\varphi', \vec{A}')$ satisfies Eq. (6.15).

→

$$\boxed{\begin{aligned} &\text{Eq. (6.10c)} \quad \Leftrightarrow \quad \left(\tfrac{1}{c^2}\partial_t^2 - \nabla^2\right)\varphi = \tfrac{4\pi}{c}\, c\rho && (6.16a)\\ &\text{Eq. (6.10d)} \quad \Leftrightarrow \quad \left(\tfrac{1}{c^2}\partial_t^2 - \nabla^2\right)\vec{A} = \tfrac{4\pi}{c}\,\vec{j} && (6.16b)\end{aligned}}$$

- Expressed in potentials in the Lorenz gauge, the inhomogeneous Maxwell equations become a set of four decoupled wave equations.
- We do not have to consider the *homogeneous* Maxwell equations in the gauge field representation because Eq. (6.12) ensures that Eqs. (6.10a) and (6.10b) are automatically satisfied.

2 | Observation: Charge $dq = \rho\, \mathrm{d}^3x$ in volume $\mathrm{d}V = \mathrm{d}^3x$ independent of inertial system:



$$\rho\, \mathrm{d}^3x = \bar{\rho}\, \mathrm{d}^3\bar{x} \quad \Rightarrow \quad \underbrace{\underbrace{\rho\, \mathrm{d}^3x}_{\text{Scalar}}\ \underbrace{\mathrm{d}x^\mu}_{4\text{—vector}}}_{4\text{—vector}} = \rho\, \mathrm{d}^3x\, \mathrm{d}t\, \frac{\mathrm{d}x^\mu}{\mathrm{d}t} = \underbrace{\frac{1}{c}\, \mathrm{d}^4x}_{\substack{\text{Eq. (4.23)}\\ \Downarrow \\ \text{Scalar}}}\ \underbrace{\rho\frac{\mathrm{d}x^\mu}{\mathrm{d}t}}_{\substack{\Downarrow \\ 4\text{—vector}}} \tag{6.17}$$

This suggests that charge and current density are actually components of a Lorentz 4-vector:

$$\boxed{j^\mu := \rho\frac{\mathrm{d}x^\mu}{\mathrm{d}t} = \begin{pmatrix} c\rho \\ \rho\vec{v}\end{pmatrix} = \begin{pmatrix} c\rho \\ \vec{j}\end{pmatrix} \quad \text{⁂ } \textit{4-current (density)}} \tag{6.18}$$

with ⁂ *charge density* $\rho = \rho(x)$ and ⁂ *current density* $\vec{j} = \vec{j}(x) = \rho(x)\vec{v}(x)$.

- In the argument above, the trajectory $\vec{x}(t)$ in $x^\mu = (ct, \vec{x}(t))$ parametrizes the movement of the infinitesimal volume $\mathrm{d}V = \mathrm{d}^3x$ with charge $\mathrm{d}q = \rho\mathrm{d}V$; the coordinate velocity $\vec{v}(t) = \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}$ is therefore the velocity of the charge distribution at position $\vec{x}(t)$ at time t: $\vec{v}(x)$. Thus, in general, the current density $\vec{j}(x) = \rho(x)\vec{v}(x)$ depends on position and time via the charge density $\rho(x)$ and the velocity field $\vec{v}(x)$.
- That the charge density ρ is *not* a Lorentz scalar is intuitively clear as it is defined as charge per volume. Volumes, however, are clearly not Lorentz invariant because they are Lorentz contracted. Since the charge (not the charge density!) *is* Lorentz invariant (this is an observational fact), the ratio of charge by volume must change under boosts.

3 | Eq. (6.18) and Eq. (6.16) suggest the compact notation

$$\boxed{\left.\begin{array}{l}\text{Eq. (6.16a)}\\ \text{Eq. (6.16b)}\end{array}\right\} \quad \partial^2 A^\mu = \frac{4\pi}{c} j^\mu \quad \text{(IME in LG)}} \tag{6.19}$$

Remember that $\partial^2 = \Box = \frac{1}{c^2}\partial_t^2 - \nabla^2$.

with

$$\boxed{A^\mu := \begin{pmatrix}\varphi \\ \vec{A}\end{pmatrix} \quad \text{⁂ } \textit{4-potential}} \tag{6.20}$$

The covariant components of the gauge field are $A_\mu = (\varphi, -\vec{A})$.

The transformation of the 4-potential must be that of a Lorentz 4-vector:

$$\left.\begin{array}{r}\bar{\partial}^2 = \partial^2 \ : \ \text{Scalar [Eq. (4.36b)]}\\ \bar{j}^\mu = \Lambda^\mu{}_\nu j^\nu \ : \ \text{4-vector [Eq. (6.18)]}\end{array}\right\} \quad \rightarrow \quad \bar{A}^\mu = \Lambda^\mu{}_\nu A^\nu \ : \ \text{4-vector} \tag{6.21}$$

With this transformation, the Maxwell equations in their simple formulation Eq. (6.19) are *manifestly* Lorentz covariant:

$$\partial^2 A^\mu = \frac{4\pi}{c} j^\mu \quad \xrightarrow{K \xrightarrow{R,\vec{v},s,\vec{b}} \bar{K}} \quad \bar{\partial}^2 \bar{A}^\mu = \frac{4\pi}{c} \bar{j}^\mu \tag{6.22}$$

4 | We can now rewrite our previous equations in tensor notation:

i | The Lorenz gauge condition can be compactly written as:

$$\boxed{\partial A \equiv \partial_\mu A^\mu = 0 \quad \text{(Lorenz gauge)}} \tag{6.23}$$

→ The Lorenz gauge is Lorentz invariant

Note: The Lorenz gauge is named after ↑ *Ludvig Lorenz*; by contrast, the Lorentz transformation is named after ↑ *Hendrik Lorentz*. Thus: *The Lorenz gauge* (no "t") *is Lorentz invariant.*

ii | The continuity equation also becomes very simple (and Lorentz covariant):

$$\boxed{\partial j \equiv \partial_\mu j^\mu = 0 \quad \text{(Continuity equation)}} \tag{6.24}$$

iii | The gauge transformation can be written as follows:

$$\boxed{A'^\mu = A^\mu - \partial^\mu \lambda \quad \text{(Gauge transformation)}} \tag{6.25}$$

Recall that $\partial^\mu = (\frac{1}{c}\partial_t, -\nabla)$.

5 | Let us summarize our findings so far:

$$\left.\begin{array}{rl} \text{Maxwell equations}: & \partial^2 A^\mu = \frac{4\pi}{c} j^\mu \\ \text{Lorenz gauge}: & \partial A = 0 \\ \text{Continuity equation}: & \partial j = 0 \end{array}\right\} \xrightarrow{K \xrightarrow{\Lambda} \bar{K}} \left\{\begin{array}{l} \bar{\partial}^2 \bar{A}^\mu = \frac{4\pi}{c} \bar{j}^\mu \\ \bar{\partial} \bar{A} = 0 \\ \bar{\partial} \bar{j} = 0 \end{array}\right. \tag{6.26}$$

→ Electrodynamics satisfies Einstein's principle of Special Relativity SR

- In contrast to Newtonian mechanics, electrodynamics was a relativistic theory all along and there was no need to modify it. It's Lorentz covariance was simply not manifest and required a bit of work to unveil.
- The treatment above relies on (1) expressing the Maxwell equations in terms of the gauge fields and (2) choosing a particular gauge (the Lorenz gauge). While this is mathematically legit (and not restrictive), it would be nice to have manifestly Lorentz covariant expressions (1) withou fixing a gauge and (2) in terms of the physically observable fields $\vec{E}$ and $\vec{B}$.

To achieve both goals, we first need a new tensorial quantity:

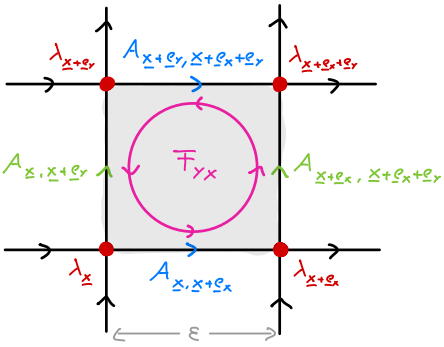
6 | Field strength tensor:

i | Motivation: We are looking for the simplest field that …

- …is gauge-invariant (i.e., has a physical interpretation).
- …is Lorentz covariant (i.e., can be used to construct Lorentz covariant equations).

ii | ◁ Discretized spacetime on a (hypercubic) lattice (here we consider the xy-plane):

- The gauge field A^μ lives on *edges* in μ-direction.
- The gauge transformation λ lives on *vertices* of the lattice.



→ Discretized gauge transformation:

$$A'_{\boldsymbol{x},\boldsymbol{x}+\boldsymbol{e}_\mu} = A_{\boldsymbol{x},\boldsymbol{x}+\boldsymbol{e}_\mu} + \underbrace{\tfrac{1}{\varepsilon}\left(\lambda_{\boldsymbol{x}+\boldsymbol{e}_\mu} - \lambda_{\boldsymbol{x}}\right)}_{\sim\partial_\mu\lambda} \tag{6.27}$$

→ Sums along paths P transform non-trivially only at their "start site" $\boldsymbol{s}$ and "end site" $\boldsymbol{e}$:

$$\sum_{e\in P} A'_e = \sum_{e\in P} A_e + \tfrac{1}{\varepsilon}\left(\lambda_{\boldsymbol{e}} - \lambda_{\boldsymbol{s}}\right) \tag{6.28}$$

Edges e are pairs of adjacent lattice sites, e.g., $e = (\boldsymbol{x}, \boldsymbol{x} + \boldsymbol{e}_x)$ with lattice vector $|\boldsymbol{e}_x| = \varepsilon$.

→ Sums $\sum_{e\in L} A_e$ along *closed loops* L are *gauge-invariant* (because $\boldsymbol{s} = \boldsymbol{e}$)!

→ Smallest gauge-invariant loop (= loop around a single face $f = yx$):

$$F_{yx} := A_{\boldsymbol{x},\boldsymbol{x}+\boldsymbol{e}_x} + A_{\boldsymbol{x}+\boldsymbol{e}_x,\boldsymbol{x}+\boldsymbol{e}_x+\boldsymbol{e}_y} - A_{\boldsymbol{x}+\boldsymbol{e}_y,\boldsymbol{x}+\boldsymbol{e}_x+\boldsymbol{e}_y} - A_{\boldsymbol{x},\boldsymbol{x}+\boldsymbol{e}_y} \tag{6.29a}$$

$$= \left(A_{\boldsymbol{x},\boldsymbol{x}+\boldsymbol{e}_x} - A_{\boldsymbol{x}+\boldsymbol{e}_y,\boldsymbol{x}+\boldsymbol{e}_x+\boldsymbol{e}_y}\right) - \left(A_{\boldsymbol{x},\boldsymbol{x}+\boldsymbol{e}_y} - A_{\boldsymbol{x}+\boldsymbol{e}_x,\boldsymbol{x}+\boldsymbol{e}_y+\boldsymbol{e}_x}\right)$$

$$\xrightarrow{\varepsilon\to 0} \partial_y A_x - \partial_x A_y \tag{6.29b}$$

iii | This motivates the definition:

$$\boxed{F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu \quad \text{⁂ Field strength tensor (FST)}} \tag{6.30a}$$

$$\overset{6.12}{\underset{6.20}{=}} \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}_{\mu\nu} \tag{6.30b}$$

Details: ➲ Problemset 7

→ $F_{\mu\nu}$ is a $(0,2)$ Lorentz tensor

- The FST is gauge-invariant by construction. You can also check this by applying the gauge transformation Eq. (6.25).
- It is easy to see that the FST has the following properties:

$$\text{Antisymmetry:} \quad F^{\mu\nu} = -F^{\nu\mu} \tag{6.31a}$$

$$\text{Tracelessness:} \quad F^{\mu}{}_{\mu} = g_{\mu\nu}F^{\mu\nu} = 0 \tag{6.31b}$$

- ¡! When we write "E_x", we refer to the x-component of the original electric field $\vec{E}$ as it occurs in the Maxwell equations Eq. (6.10). In this context, an expression like E^x does not make sense since $\vec{E}$ is not a 4-vector but the component of a rank-2 tensor.

iv | Using that $\varepsilon^{\mu\nu\alpha\beta}$ is a Lorentz pseudo-tensor [recall Eq. (4.41)], we can define:

$$\boxed{\tilde{F}^{\mu\nu} := \tfrac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} \quad \text{⁂ Dual field strength tensor (DFST)}} \tag{6.32a}$$

$$\overset{3.41}{\underset{6.30}{=}} \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}_{\mu\nu} \tag{6.32b}$$

Details: ➲ Problemset 7

→ $\tilde{F}^{\mu\nu}$ is a $(2,0)$ *pseudo* Lorentz tensor

- The dual field-strength tensor will be useful below.
- $\tilde{F}^{\mu\nu}$ is obtained from $F^{\mu\nu}$ (*contravariant*!) by the substitution $\vec{E} \mapsto \vec{B}$ and $\vec{B} \mapsto -\vec{E}$. [Just like in vacuum the homogeneous Maxwell equations Eqs. (6.10a) and (6.10b) transform into the "inhomogeneous" ones Eqs. (6.10c) and (6.10d)].

7 | Transformation of the electromagnetic field:

The field strength tensor Eq. (6.30) has the useful properties that (1) we know how it transforms under Lorentz transformations, and (2) we know how it relates to the observable fields $\vec{E}$ and $\vec{B}$. Hence we can use it to derive the transformation of the electromagnetic field when transitioning from one inertial system to another.

i | The (contravariant) FST transforms under a Lorentz transformation Λ as follows:

$$\underbrace{\bar{F}^{\mu\nu}(\bar{x})}_{\{\bar{E}_i(\bar{x}),\,\bar{B}_i(\bar{x})\}} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\underbrace{F^{\alpha\beta}(x)}_{\{E_i(x),\,B_i(x)\}} \tag{6.33}$$

Here it is $F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta}$ as usual.

ii | $\lhd$ Boost $\Lambda_{\vec{v}}$ [Eq. (4.10)]:

$$\boxed{\begin{aligned}\bar{\vec{E}}(\bar{x}) &\mathrel{\hat{=}} \gamma\left[\vec{E}(x) + \tfrac{1}{c}\vec{v}\times\vec{B}(x)\right] - (\gamma-1)\frac{\vec{v}\cdot\vec{E}(x)}{v^2}\vec{v} \qquad (6.34a)\\ \bar{\vec{B}}(\bar{x}) &\mathrel{\hat{=}} \gamma\left[\vec{B}(x) - \tfrac{1}{c}\vec{v}\times\vec{E}(x)\right] - (\gamma-1)\frac{\vec{v}\cdot\vec{B}(x)}{v^2}\vec{v} \qquad (6.34b)\end{aligned}}$$

with $x^{\mu} = (\Lambda_{-\vec{v}})^{\mu}{}_{\nu}\,\bar{x}^{\nu}$.
¡! Note that on the left-hand side the arguments are $\bar{x}$ and on the right-hand side x!

→ Electric and magnetic fields "mix" under boosts!

- Please appreciate what we showed: If you start from Maxwell Eq. (6.10) and perform an arbitrary Lorentz boost $\bar{x}^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\nu}$, transforming the derivatives as $\bar{\partial}_{\mu} = \Lambda_{\mu}{}^{\nu}\partial_{\nu}$, you obtain a set of horribly looking PDEs. But if you recombine the equations appropriately, group the terms according to Eq. (6.34) and define the new fields $\bar{\vec{E}}(\bar{x})$, $\bar{\vec{B}}(\bar{x})$, the equations look again like Eq. (6.10), only with bars over coordinates and fields.

  You *could* show this directly, without ever introducing the gauge field A^{μ} and without using the machinery of tensor calculus (this is what Einstein did for a boost in z-direction in his 1905 paper "Zur Elektrodynamik bewegter Körper" [9]); but hopefully you agree that our more advanced route (using the gauge field and tensor calculus) is a more elegant approach.

- Because of our motivation from Einstein's principle of Special Relativity SR, we frame our discussion in the terminology of *passive* transformations (= coordinate transformation): The same electromagnetic field that looks like $\vec{E}(x)$, $\vec{B}(x)$ in an inertial system K looks like $\bar{\vec{E}}(\bar{x})$, $\bar{\vec{B}}(\bar{x})$ in another system $\bar{K}$.

  Because we showed that the Maxwell equations satisfy SR, they have exactly the same form in $\bar{K}$ as in K. This, however, allows you to interpret the transformation *actively*: If you are given a solution of Maxwell equations $\vec{E}(x)$, $\vec{B}(x)$, then, for any $\vec{v}$, the *new* functions $\bar{\vec{E}}(\bar{x})$, $\bar{\vec{B}}(\bar{x})$ defined by Eq. (6.34) and $x^{\mu} = (\Lambda_{-\vec{v}})^{\mu}{}_{\nu}\,\bar{x}^{\nu}$ are again solutions (in the same coordinates). This shows that the Lorentz group is (part of) the invariance group of the PDE system Eq. (6.10) we call Maxwell equations (just like the Galiei group was an invariance group of Newton's equation, recall Section 1.2).