↓**Lecture 11** [09.01.24]

12 | ⊲ Non-relativistic limit:

$$E = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} \approx \underbrace{mc^2}_{\substack{\text{Rest}\\\text{energy}}} + \underbrace{\frac{1}{2}mv^2}_{\substack{\text{Newtonian}\\\text{kinetic}\\\text{energy}}} + \mathcal{O}\left(\beta^4\right) \tag{5.35}$$

This shows again that the correspondence principle is satisfied: For small velocities compared to c, the kinetic energy of Newtonian mechanics is (up to a constant shift given by the rest energy) a good proxy for the true energy of the particle.

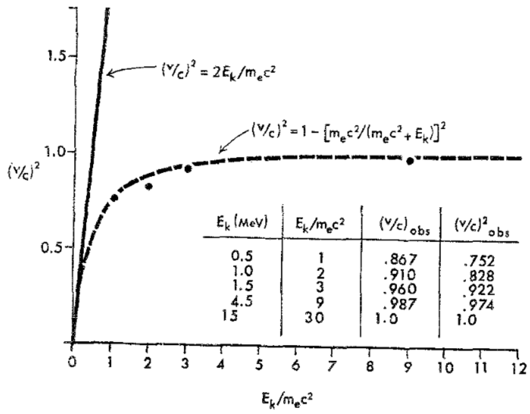
13 | The *kinetic energy* is: $E_{\text{kin}} = E - E_0 = E - mc^2$

→ The velocity of a relativistic particle as a function of its *kinetic energy* is:

$$\beta^2 = \left(\frac{v}{c}\right)^2 = 1 - \left[\frac{mc^2}{E_{\text{kin}} + mc^2}\right]^2 \xrightarrow{\beta \ll 1} \frac{2E_{\text{kin}}}{mc^2} \tag{5.36}$$

Note that in the non-relativistic limit it is $E_{\text{kin}} \ll mc^2$.

This velocity dependence has been confirmed experimentally to high precision; for example with accelerated electrons [33] (see Refs. [33, 34] for more technical details):



→ The relativistic energy relation Eq. (5.24) is correct ☺☺

14 | <u>Massless particles:</u>

So far we considered only particles with non-vanishing mass $m \neq 0$. The definition of the momentum Eq. (5.1) and the relativistic energy Eq. (5.24) cannot be directly applied to particles without mass. However:

i | ⊲ Eq. (5.26) with $m \to 0$:

$$\boxed{E = |\vec{p}|c \quad \text{(linear dispersion)}} \tag{5.37}$$

⊲ Eq. (5.4) with $m \to 0$:

$$\boxed{p^2 = 0 \ \text{(light-like)} \quad \Rightarrow \quad p^\mu = \begin{pmatrix} |\vec{p}| \\ \vec{p} \end{pmatrix}} \tag{5.38}$$

¡! We take this as the *definition* of the 4-momentum for massless particles (it is the only definition that is consistent with $p^\mu = mu^\mu$ in the limit of vanishing mass). Note that there is no finite 4-velocity u^μ associated to massless particles.

ii | The fact that p^μ becomes light-like for massless particles already suggests that they move with the speed of light. We can verify this:

$$\left.\begin{aligned} E &= \gamma_v mc^2 \\ \vec{p} &= \gamma_v m\vec{v} \end{aligned}\right\} \quad\Rightarrow\quad E = |\vec{p}|\frac{c^2}{v} \quad \xrightarrow[\text{Eq. (5.37)}]{m\to 0} \quad |\vec{p}|\,c \tag{5.39}$$

This limit is only consistent if $v \to c$ for $m \to 0$:

$$\boxed{\text{All particles with vanishing mass move with the speed of light.}} \tag{5.40}$$

- Examples: Photons, Gravitons (if they exist)
- Massless particles do not have a rest frame.

  You would need a boost with $v = c$ to reach such a frame; but such boosts are not defined (because the Lorentz factor diverges in this limit).
- ¡! The relativistic energy $E = \gamma_v mc^2$ holds only for massive particles. For massless particles it does *not* follow $E = 0$ but rather $E = |\vec{p}|c \neq 0$. So photons do have energy and momentum, but no mass (neither rest- nor any other type of mass). You are also not allowed to use the "forbidden" equation $E = m_r c^2$ and declare $m_r = E/c^2 = |\vec{p}|/c$ as the "dynamic mass" of the photon because (1) we argued above that this concept is not as useful as it sounds, and (2) you only renamed momentum, so what's the point. And if you are afraid that later – in GENERAL RELATIVITY– our photons will not be deflected by stars or sucked into black holes because they "have no mass": I assure you, they will; they have energy and momentum, that's enough.
- This demonstrates why the "speed of light" is sort of a misnomer in this context, and we should have stuck to our $v_{\max}$ (but then all our equations would look different from the literature). Then it would be conceptually clear that *every* particle with vanishing rest mass "runs into" the universal speed limit $v_{\max}$.

5.3. Action principle and conserved quantities

In this section we study a more formal (and more versatile) approach to describe the dynamics of relativistic systems, namely in terms of the Lagrangian and the action. We do this for the free particle (no force!) and consider electromagnetic forces in the next Chapter 6.

1 | Action of free massive particle:

i | $\triangleleft$ Trajectory γ parametrized by
$x^\mu = x^\mu(\lambda)$ with $\lambda \in [\lambda_a, \lambda_b]$ and $x^\mu(\lambda_a) = a^\mu$, $x^\mu(\lambda_b) = b^\mu$

Remember the characteristic property of the trajectory of a free particle (Section 2.4):
The proper time (= Minkowski distance) is *maximized* along the trajectory!

$$\boxed{\to \text{Action:} \quad S[\gamma] := \alpha \int_\gamma \mathrm{d}s = \alpha \int_{\lambda_a}^{\lambda_b} \sqrt{\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}\,\mathrm{d}\lambda} \tag{5.41}$$

with $\dot{x}^\mu \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}$.

The prefactor α is undetermined so far (→ *next step*).

¡! The parameter λ has no physical interpretation in this formulation as this action is reparametrization invariant (→ *Section 5.4*).

ii | Correspondence principle $\rightarrow \alpha = -mc$

To determine the parameter α, consider the non-relativistic limit of the Lagrangian in coordinate time parametrization $\lambda = t$:

$$\tilde{L} = \alpha\sqrt{c^2 - \dot{\vec{x}}^2} = \alpha c\sqrt{1 - \frac{v^2}{c^2}} \xrightarrow{\beta \ll 1} L \approx \alpha c - \underbrace{\frac{\alpha v^2}{2c}}_{\stackrel{!}{=}\frac{1}{2}mv^2} \tag{5.42}$$

The non-relativistic limit yields – up to a constant that doesn't change the equations of motion – the Lagrangian with Newtonian kinetic energy if we set $\alpha = -mc$.

iii | Lagrangian:

$$L(x^\mu, \dot{x}^\mu) = -mc\sqrt{\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} = -mc\sqrt{\dot{x}_\mu\dot{x}^\mu} \tag{5.43}$$

- ¡! This Lagrangian is only valid for *massive* particles.
- The Lagrangian Eq. (5.43) is fully specified as is; there is no need to fix a specific parametrization. In this form, the Lagrangian [more precisely: the action Eq. (5.41)] has a gauge symmetry: the parametrization λ is arbitrary (→ *Section 5.4*).
- On the contrary, if you fix a parametrization (= fix a gauge), e.g., by identifying λ with the coordinate time $\lambda = t \equiv x^0/c$ ("static gauge") or the proper time $\lambda = \tau$ ("proper time gauge"), you obtain different (but physically equivalent) Lagrangians which have no longer a gauge symmetry:

$$\lambda \stackrel{!}{=} t \quad\Leftrightarrow\quad c\lambda \stackrel{!}{=} x^0 \quad\Rightarrow\quad \tilde{L}_t(\vec{x}, \dot{\vec{x}}) = -mc^2\sqrt{1 - \dot{\vec{x}}^2/c^2}\,, \tag{5.44a}$$

$$\lambda \stackrel{!}{=} \tau \quad\Leftrightarrow\quad \dot{x}^\mu\dot{x}_\mu \stackrel{!}{=} c^2 \quad\Rightarrow\quad \tilde{L}_\tau(x^\mu, \dot{x}^\mu) = -mc^2\,. \tag{5.44b}$$

We denote gauge-fixed Lagrangians by $\tilde{L}$ and the gauge-invariant Lagrangian Eq. (5.43) by L. In the following we often work with the latter and choose specific parametrizations at the end of our calculations to express results in known quantities.

2 | Euler-Lagrange equations:

$$\delta S \stackrel{!}{=} 0 \quad\Rightarrow\quad \frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial L}{\partial \dot{x}^\sigma} - \underbrace{\frac{\partial L}{\partial x^\sigma}}_{=0} = 0 \quad\Rightarrow\quad \frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{-mc\dot{x}_\sigma}{\sqrt{\dot{x}_\mu\dot{x}^\mu}} = 0 \tag{5.45}$$

These are 4 differential equations ($\sigma = 0, 1, 2, 3$)!

→ Equations of motion in the "proper time gauge" $\lambda = \tau$ [where $\dot{x}_\mu\dot{x}^\mu = u^2 = c^2$]:

$$m\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = 0 \tag{5.46}$$

This is Eq. (5.6) for vanishing 4-force ☺

3 | ⊲ Action in "static gauge" $\lambda = t = x^0/c$:

$$S[\gamma] \overset{\lambda=\frac{x^0}{c}}{\equiv} \tilde{S}_t[\vec{x}(t)] = \int_{t_a}^{t_b} \tilde{L}_t(\vec{x}, \dot{\vec{x}})\, dt = -mc^2 \int_{t_a}^{t_b} \sqrt{1 - \frac{\dot{\vec{x}}^2}{c^2}}\, dt \tag{5.47}$$

i | Canonical momenta ($\vec{v} = \dot{\vec{x}}$):

$$\vec{p} = \frac{\partial \tilde{L}_t}{\partial \vec{v}} = \frac{m\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{5.48}$$

This is the expression for the relativistic 3-momentum Eq. (5.3) we found before, now derived as the canonical momentum of a Lagrangian.

ii | Hamiltonian:

$$\tilde{H}_t = \vec{p} \cdot \vec{v} - \tilde{L}_t \stackrel{\circ}{=} \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} = cp^0 \overset{5.26}{=} c\sqrt{\vec{p}^2 + m^2c^2} \tag{5.49}$$

This is just the relativistic energy Eq. (5.24) we found before, now derived from a Lagrangian.

- ⊲ Non-relativistic limit:

$$\tilde{H}_t = mc^2 \sqrt{1 + \frac{\vec{p}^2}{m^2c^2}} \overset{\frac{\vec{p}^2}{2m} \ll mc^2}{\approx} \underbrace{mc^2}_{\text{Rest energy}} + \underbrace{\frac{\vec{p}^2}{2m}}_{\text{Newtonian kinetic energy}} \tag{5.50}$$

- ¡! Contrary to the action Eq. (5.47), this Hamiltonian also makes sense for *massless* particles:

$$\tilde{H}_t \overset{m=0}{=} |\vec{p}|c \tag{5.51}$$

4 | Noether's (first) theorem:

Details: ➔ Problemset 6

x^μ cyclic → Spacetime translations $x^\mu + \delta\varepsilon^\mu$ are *continuous symmetries* of S

These transformations correspond to the inhomogeneous part of Poincaré transformations: $\bar{x}^\mu = x^\mu + a^\mu$. Every relativistic theory must have this symmetry; for field theories one obtains then four conserved currents: → *Energy momentum tensor*.

↓ *Noether's theorem* → ↓ *Conserved Noether charges* Q_μ: (set $\lambda = t$ as the coordinate time)

$$Q_\mu \equiv \left\{ \begin{array}{ll} \text{Time translation} & \Rightarrow \text{Energy } E/c \\ \text{Space translations} & \Rightarrow \text{Momentum } \vec{p} \end{array} \right\} \tag{5.52}$$

$$= -\frac{\partial L}{\partial \dot{x}^\mu} = \frac{mc\dot{x}_\mu}{\sqrt{c^2 - \vec{v}^2}} = \begin{pmatrix} \frac{1}{c}\frac{mc^2}{\sqrt{1-\beta^2}} \\ -\frac{m\vec{v}}{\sqrt{1-\beta^2}} \end{pmatrix} = p_\mu \tag{5.53}$$

- Because x^μ are cyclic coordinates, we can obtain the Noether charges directly from the Lagrangian as $\frac{\partial L}{\partial \dot{x}^\mu}$; the additional minus is conventional to connect to our definition of the 4-momentum.
- ¡! This shows that our definition of the 4-momentum is consistent, and the identification of its time-component p^0 as the total energy was correct: By definition, *energy* is the Noether charge that corresponds to translation invariance in *time*. Similarly, *momentum* is the charge for translation invariance in *space*.

5 | Noether charges for *homogeneous* Lorentz transformations?

Any relativistic theory is also invariant under (proper orthochronous) Lorentz transformations, $\bar{x}^\mu = \Lambda^\mu{}_\nu x^\mu$; for these there must exist additional conserved Noether charges:

Infinitesimal Lorentz transformations $x^\mu + \delta\varepsilon^\mu{}_\nu x^\nu$ are *continuous symmetries* of S

The infinitesimal transformation is antisymmetric: $\delta\varepsilon^\mu{}_\nu = -\delta\varepsilon_\nu{}^\mu$, ➔ Problemset 5.

$\overset{\circ}{\rightarrow}$ Conserved Noether charges:

$$\text{⁂ } \textit{Angular momentum (tensor):} \quad L^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu \tag{5.54}$$

This is an example of an antisymmetric $(2,0)$ Lorentz tensor.

Proof: ➔ Problemset 6

i | ⊲ Spatial components:

$$\left.\begin{aligned} L^{23} &= x^2p^3 - x^3p^2 = l_1 \\ L^{31} &= x^3p^1 - x^1p^3 = l_2 \\ L^{12} &= x^1p^2 - x^2p^1 = l_3 \end{aligned}\right\} \quad \text{with 3-angular momentum } \vec{l} = \vec{x} \times \vec{p}. \tag{5.55}$$

→ 3-angular momentum $\vec{l}$ is *not* (part of a) Lorentz *vector* but of a $(2,0)$ *tensor*!

It is not surprising that invariance under spatial rotations $\mathrm{SO}(3) \subset \mathrm{O}(1,3)$ implies angular momentum conservation.

ii | ⊲ Mixed components:

$$\begin{aligned} L^{10} &= x^1\gamma_v mc - ctp^1 = cn_1 \\ L^{20} &= x^2\gamma_v mc - ctp^2 = cn_2 \\ L^{30} &= x^3\gamma_v mc - ctp^3 = cn_3 \end{aligned} \tag{5.56}$$

with ⁂ *dynamic mass moment*

$$\vec{n} := m\gamma_v\left(\vec{x} - t\vec{v}\right) = \frac{E}{c^2}\vec{x} - t\,\vec{p} = \text{const}\,. \tag{5.57}$$

This is the relativistic version of the ↓ *center-of-mass theorem.*

The *center of mass* (COM) is now the *center of energy* (COE). Since $\vec{n}$ (and E) is conserved, we can set $t = 0$ to find $\vec{n} = E/c^2\vec{x}_0$, which is the initial center of energy of the system (times E/c^2).

For many particles this is slightly less trivial: One finds analogously the conserved quantity

$$\vec{N} = \sum_i \vec{n}_i = \sum_i \left(\frac{E_i}{c^2}\vec{x}_i - t\,\vec{p}_i\right) = \text{const}\,. \tag{5.58}$$

Division by the total (also conserved) energy $E = \sum_i E_i$ yields

$$\vec{X}_{\mathrm{COE}}(t) := \frac{\sum_i E_i \vec{x}_i}{\sum_i E_i} = t\,\frac{c^2 \vec{P}}{E} + \mathrm{const} \equiv t\,\vec{V}_{\mathrm{COE}} + \mathrm{const} \tag{5.59}$$

with the total 3-momentum $\vec{P} = \sum_i \vec{p}_i$. Thus the ⁂ *center of energy* $\vec{X}_{\mathrm{COE}}$ moves in a straight line with constant velocity $\vec{V}_{\mathrm{COE}}$. Note that the center of energy becomes the Newtonian center of mass in the non-relativistic limit where $E_i \approx E_{i,0} = m_i c^2$.

6 | ∢ Multiple particles (covariantly coupled by fields):

The above arguments can be directly generalized to many (non-interacting) particles. This immediately yields the sum of the 4-momenta of these particles as conserved quantity. Interactions between the particles must be covariantly mediated by fields – which also carry 4-momentum (→ *Chapter 6*):

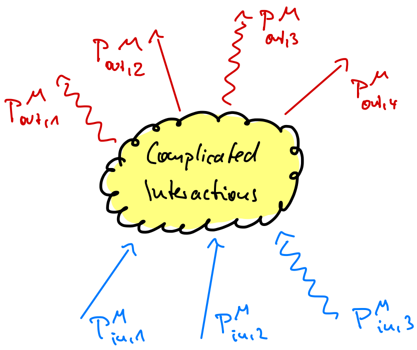
Conserved Noether charge:

⁂ *Total 4-momentum*: $$P^\mu := \sum_i p_i^\mu + p^\mu_{\mathrm{Fields}} \tag{5.60}$$

with

- p_i^μ the 4-momentum of particle i, and
- p^μ_{Fields} the total 4-momentum of the fields mediating the interations.

7 | ∢ Scattering process:



Long before and after the interactions play a role we can approximate the system by non-interacting particles and set $p^\mu_{\mathrm{Fields}} = 0 \rightarrow$

$$\sum_i p^\mu_{\mathrm{in},i} = \sum_j p^\mu_{\mathrm{out},j} \tag{5.61}$$

$\rightarrow$ Conservation of *energy* $(\mu = 0)$ and *momentum* $(\mu = 1, 2, 3)$

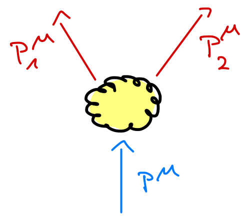
- In RELATIVITY, conservation of total energy and total momentum is combined into the conservation of 4-momentum.
- We will denote the 4-momenta of *massive* particles (solid lines) with p^μ and the 4-momenta of *massless* particles with q^μ (wiggly lines).

Examples:

i | Particle decay: ∢ Radioactive Nucleus → Nucleus 1 & Nucleus 2

→ Energy-momentum conservation:

$$\underbrace{p^\mu}_{\text{in}} = \underbrace{p_1^\mu + p_2^\mu}_{\text{out}} \tag{5.62}$$



∢ Center-of-mass frame where $\vec{p} = \vec{p}_1 + \vec{p}_2 = 0 \overset{\circ}{\rightarrow}$

$$mc^2 = m_1c^2 + E_{\text{kin},1} + m_2c^2 + E_{\text{kin},2} \tag{5.63}$$

→ Decay only possible if

$$m \geq m_1 + m_2 \tag{5.64}$$

If $E_{\text{kin},1} \neq 0$ or $E_{\text{kin},2} \neq 0$, it is $m \neq m_1 + m_2$.

→ The rest mass of composite objects is *not additive*.

Composite objects also contain binding energy (potential energy) which contributes to the rest mass of the object.

$\overset{\circ}{\rightarrow}$

$$E_{\text{kin},1} = \frac{(m - m_1)^2c^2 - m_2^2c^2}{2m} \tag{5.65}$$

In the COM frame, the kinetic energy of the two decay products is constant and depends only on the masses of the particles. So if you find a non-trivial energy distribution for the products of a decay process, there must at least three particles be produced (of which you might not be able to detect all). This is how the neutrino was predicted by Pauli from the decay of the neutron: $n \to p + e^- + \bar{\nu}_e$.

ii | Particle creation:

Note that a single massless (light-like) particle (like a photon) cannot decay into two massive (time-like) particles because $(p_1 + p_2)^2 = q^2 = 0$ cannot be solved if $p_i^2 = m_i^2c^2 > 0$.

Indeed (we set $c = 1$): With the ↓ *Cauchy-Schwarz inequality* we find

$$m_1m_2 + \vec{p}_1 \cdot \vec{p}_2 \leq \sqrt{m_1^2 + \vec{p}_1^2}\sqrt{m_2^2 + \vec{p}_2^2} = p_1^0 p_2^0 \tag{5.66a}$$

$$\Rightarrow \quad 0 < m_1m_2 \leq p_1 \cdot p_2 \tag{5.66b}$$

so that for arbitrary m_1 and m_2 (particle creation: $q^\mu = p_1^\mu + p_2^\mu$)

$$(p_1 + p_2)^2 = m_1^2 + m_2^2 + 2p_1 \cdot p_2 > 0 \quad \Rightarrow \quad \text{Time-like} \tag{5.67}$$

Furthermore, for $m_1 = m_2$ (scattering: $p_1^\mu - p_2^\mu = q^\mu$):

$$(p_1 - p_2)^2 = m_1^2 + m_2^2 - 2p_1 \cdot p_2 \tag{5.68a}$$

$$\leq m_1^2 + m_2^2 - 2m_1m_2 \overset{m_1=m_2}{=} 0 \quad \xrightarrow{p_1 \neq p_2} \quad \text{Space-like} \tag{5.68b}$$

(For the Cauchy-Schwarz inequality, *equality* holds iff the two vectors are linearly dependent; for $m_1 = m_2$ this is only possible if $p_1 = p_2$, i.e., in the trivial case of no scattering.)

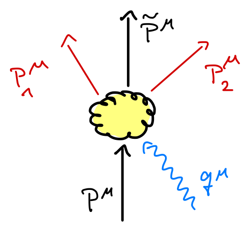
Eq. (5.67) shows that two particles (of arbitrary masses) can never annihilate into a single photon, and, vice versa, a single photon can never create a pair of massive particles. This is reason why we need an additional (heavy) nucleus for the creation of a particle & antiparticle pair from a photon.

By contrast, Eq. (5.68) tells us that a single massive particle cannot emit or absorb a single photon *if it cannot change its mass* (i.e., has no different energy states). This is true for free elementary particles like electrons (an electron cannot be excited, it always has the same mass). Thus a free electron cannot emit a single photon. If the massive particle in question *has* different internal energy states (and therefore the two masses m_1 and m_2 can be different), this argument does not hold. This is why atoms can spontaneously emit or absorb single photons.

$\sphericalangle$ Photon (+Nucleus) → Electron & Positron (+Nucleus)

→ Energy-momentum conservation:

$$P_{\text{in}}^\mu \equiv \underbrace{q^\mu + p^\mu}_{\text{in}} = \underbrace{p_1^\mu + p_2^\mu + \tilde{p}^\mu}_{\text{out}} \equiv P_{\text{out}}^\mu \tag{5.69}$$



With the mass M of the nucleus and the momentum/energy $|\vec{q}| = E_\gamma/c$ of the incoming photon, we find

$$\underbrace{\left(\frac{E_\gamma + Mc^2}{c}\right)^2 - \left(\frac{E_\gamma}{c}\right)^2}_{\text{Rest frame of nucleus}} = P_{\text{in}}^2 \overset{!}{=} P_{\text{out}}^2 = \underbrace{\left(\frac{E_{\text{Nuc}} + E_{e^-} + E_{e^+}}{c}\right)^2}_{\text{COM frame of system}} \tag{5.70}$$

where the right hand side was evaluated in the COM frame with $\vec{P}_{\text{out}} = \vec{0}$ and the left hand side in the rest frame of the nucleus (which is allowed since $P^2 = P^\mu P_\mu$ is a Lorentz scalar).

Please appreciate the subtlety of this evaluation: The 4-momentum conservation Eq. (5.69) is Lorentz *covariant*. Therefore you *cannot* evaluate the left hand side P_{in}^μ in one inertial system and the right hand side P_{out}^μ in another. However, in any inertial system Eq. (5.69) implies $P_{\text{in}}^2 = P_{\text{out}}^2$ where left and right hand side are now Lorentz *invariant*; hence you can evaluate the two sides in *different* inertial systems.

$\overset{\circ}{\to}$ Threshold for particle creation:

$$E_{\gamma,\text{min}} = 2m_e^2c^2\left(1 + \frac{m_e}{M}\right) > 2m_ec^2 \tag{5.71}$$

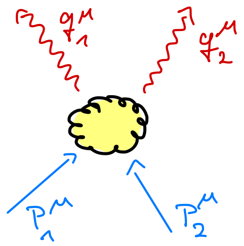
The threshold follows for vanishing kinetic energy of the products in the COM frame.

The threshold energy is larger than twice the rest energy of the electron $2m_ec^2$ (the positron has the same mass as the electron) because the scattering products necessarily aquire kinetic energy in the initial rest frame of the nucleus (to carry the momentum of the photon).

iii | <u>Annihilation:</u> $\sphericalangle$ Electron & Positron → Photon & Photon

→ Energy-momentum conservation:

$$P_{\text{in}}^{\mu} \equiv \underbrace{p_1^{\mu} + p_2^{\mu}}_{\text{in}} = \underbrace{q_1^{\mu} + q_2^{\mu}}_{\text{out}} \equiv P_{\text{out}}^{\mu} \tag{5.72}$$



⊲ COM frame:

$$P_{\text{in}}^{\mu} = \begin{pmatrix} E_{e^-}/c \\ \vec{p} \end{pmatrix} + \begin{pmatrix} E_{e^+}/c \\ -\vec{p} \end{pmatrix} = \begin{pmatrix} |\vec{q}| \\ \vec{q} \end{pmatrix} + \begin{pmatrix} |\vec{q}| \\ -\vec{q} \end{pmatrix} = P_{\text{out}}^{\mu} \tag{5.73}$$

Using that electron and positron have the same mass m_e, we find for the energy of the emitted photons:

$$E_\gamma = c\sqrt{\vec{p}^2 + m_e^2 c^2} \tag{5.74}$$

Note that the individual rest massess of particles in scattering processess are not conserved: $p_1^2 = p_2^2 = m_e^2 c^2 > 0$ for the incoming electron and the positron, but $q_1^2 = q_2^2 = 0$ for the outgoing photons. The rest mass of the *composite system* remains the same, though. In particular, the two photons together have the same rest mass as the electron-positron system before: $P_{\text{out}}^2 = P_{\text{in}}^2 = 4(\vec{p}^2 + m_e^2 c^2) > 0$.

→ The rest masses of individual particles are *not conserved*.

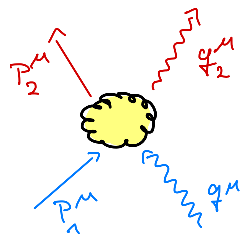
iv | Compton scattering: ⊲ Electon & Photon → Electron & Photon

Details: ➔ Problemset 6

Compton scattering is an example of ↓ *elastic scattering* where the total kinetic energy is conserved and the rest energies of in- and outgoing particles remains the same.

→ Energy-momentum conservation:

$$\underbrace{q_1^{\mu} + p_1^{\mu}}_{\text{in}} = \underbrace{q_2^{\mu} + p_2^{\mu}}_{\text{out}} \tag{5.75}$$

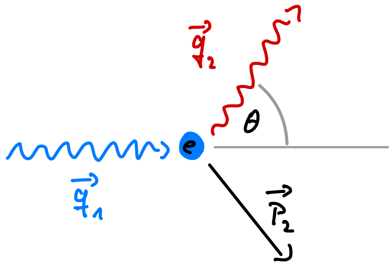


With $q_1^2 = q_2^2 = 0$ and $p_1^2 = p_2^2 = m_e^2 c^2$ one finds:

$$\underbrace{E_1 E_2/c^2(1-\cos\theta)}_{\text{Rest frame of } e^-} \stackrel{\circ}{=} \underbrace{q_1 \cdot q_2 = p \cdot (q_1 - q_2)}_{\text{Lorentz invariant}} \stackrel{\circ}{=} \underbrace{m_e c(E_1/c - E_2/c)}_{\text{Rest frame of } e^-} \tag{5.76a}$$

$$\Rightarrow \quad \frac{1}{E_2} - \frac{1}{E_1} = \frac{1}{m_e c^2}(1-\cos\theta) \tag{5.76b}$$

Here the left and right hand sides are evaluated in the rest frame of the electron: $p_1^\mu = (m_e c, \vec{0})^T$; θ is the angle between incoming and outgoing photon (scattering angle):



With the photon energy $E_i = hc/\lambda_i$ we find the change in wavelength due to scattering:

$$\Delta\lambda = \lambda_2 - \lambda_1 = \underbrace{\frac{h}{m_e c}}_{\lambda_e}(1 - \cos\theta) \tag{5.77}$$

with ⁂ *Compton wavelength* λ_e of the electron.

- With Compton scattering one can measure the Compton wavelength of the electron and thereby determine the Planck constant h.
- Because the Compton wavelength is the natural length scale associated to a massive quantum particle, it appears in many field equations of relativistic quantum mechanics (Klein-Gordon equation, Dirac equation, …).

5.4. ‡ Reparametrization invariance

The action of the free relativistic particle Eq. (5.41) has the peculiar property of "reparametrization invariance", a feature that plays an important role in GENERAL RELATIVITY, and is also crucial for the quantization of the relativistic string in string theory (↑ *Nambu-Goto action*).

1 | $\sphericalangle$ Trajectory γ parametrized by $x^\mu(\lambda)$ for $\lambda \in [\lambda_a, \lambda_b]$.

$\sphericalangle$ Diffeomorphism $\varphi : [\lambda_a, \lambda_b] \to [\lambda_a, \lambda_b]$ with $\lambda_{a/b} = \varphi(\lambda_{a/b})$ and write $\tilde{\lambda} = \varphi(\lambda)$.

Diffeomorphism = Bijective map where both the map and its inverse are continuously differentiable.

→ Define new trajectory $\tilde{\gamma}$ via $\tilde{x}^\mu(\tilde{\lambda}) := x^\mu(\varphi^{-1}(\tilde{\lambda})) = x^\mu(\lambda)$ with $\tilde{\lambda} \in [\lambda_a, \lambda_b]$.

$\tilde{x}^\mu(\tilde{\lambda})$ is a *reparametrization* of $x^\mu(\lambda)$: $\tilde{x}^\mu$ and x^μ are *different* functions on $[\lambda_a, \lambda_b]$ that parametrize the *same* trajectory in spacetime $\mathbb{R}^{1,3}$.

→ Action of new trajectory:

$$S[\tilde\gamma] \stackrel{\text{def}}{=} -mc\int_{\lambda_a}^{\lambda_b}\sqrt{\dot{\tilde x}_\mu(\lambda)\dot{\tilde x}^\mu(\lambda)}\,\mathrm{d}\lambda \tag{5.78a}$$

Rename the dummy variable: $\lambda \to \tilde\lambda$

$$= -mc\int_{\lambda_a}^{\lambda_b}\sqrt{\dot{\tilde x}_\mu(\tilde\lambda)\dot{\tilde x}^\mu(\tilde\lambda)}\,\mathrm{d}\tilde\lambda \tag{5.78b}$$

Use $\tilde x^\mu(\tilde\lambda) = x^\mu(\lambda)$ and the chain rule

$$= -mc\int_{\lambda_a}^{\lambda_b}\sqrt{\dot x_\mu(\lambda)\frac{\mathrm{d}\lambda}{\mathrm{d}\tilde\lambda}\dot x^\mu(\lambda)\frac{\mathrm{d}\lambda}{\mathrm{d}\tilde\lambda}}\,\mathrm{d}\tilde\lambda \tag{5.78c}$$

Substitution in the integral: $\tilde\lambda = \varphi(\lambda)$

$$= -mc\int_{\lambda_a}^{\lambda_b}\sqrt{\dot x_\mu(\lambda)\dot x^\mu(\lambda)}\,\mathrm{d}\lambda \tag{5.78d}$$

$$\stackrel{\text{def}}{=} S[\gamma] \tag{5.78e}$$

→ S is invariant under diffeomorphisms on parameter space.

→ ⁂ *Reparametrization invariance (RI)*

2 | Infinitesimal generators:

i | Consider infinitesimal deformations $\varepsilon(\lambda)$ of the parametrization (i.e., $|\varepsilon(\lambda)| \ll 1$ for all λ):

$$\tilde\lambda = \varphi(\lambda) \equiv \lambda + \varepsilon(\lambda) \tag{5.79}$$

With this we find:

$$x^\mu(\lambda) \stackrel{\text{def}}{=} \tilde x^\mu(\tilde\lambda) = \tilde x^\mu(\lambda+\varepsilon(\lambda)) = \tilde x^\mu(\lambda) + \varepsilon(\lambda)\partial_\lambda\tilde x^\mu(\lambda) + \mathcal{O}(\varepsilon^2) \tag{5.80}$$

ii | The infinitesimal variation of the trajectory is:

$$\delta_\varepsilon x^\mu := \tilde x^\mu(\lambda) - x^\mu(\lambda) \tag{5.81a}$$

$$= -\varepsilon(\lambda)\partial_\lambda x^\mu(\lambda) + \mathcal{O}(\varepsilon^2) \tag{5.81b}$$

$$\equiv G_\varepsilon x^\mu + \mathcal{O}(\varepsilon^2) \tag{5.81c}$$

Note that we can replace $\tilde x^\mu$ by x^μ in linear order of ε.

→ ⁂ *Generators* of one-dimensional diffeomorphisms:

$$G_\varepsilon = -\varepsilon(\lambda)\partial_\lambda \quad \text{for arbitrary (infinitesimal) } \varepsilon(\lambda). \tag{5.82}$$

iii | We can expand $\varepsilon(\lambda)$ into a Taylor series $\varepsilon(\lambda) = \sum_n \frac{\varepsilon_n}{n!}\lambda^n$ to write

$$G_\varepsilon = \sum_n \frac{\varepsilon_n}{n!}\left(-\lambda^n\partial_\lambda\right) \equiv \sum_n \frac{\varepsilon_n}{n!}G_n\,. \tag{5.83}$$

→ Basis of generators that generate infinitesimal reparametrizations is given by

$$G_n = -\lambda^n\partial_\lambda \quad \text{for } n \in \mathbb{N}_0. \tag{5.84}$$

→ RI = Infinite-dimensional continuous symmetry group

Note that in particular $\varepsilon(\lambda)$ can be chosen such that it is non-zero only for a compact subinterval of $[\lambda_a, \lambda_b]$, i.e., reparametrization invariance is a *local* symmetry (local in parameter space).

→ RI is a gauge symmetry

3 | Conserved quantities:

You know from your course on classical mechanics that Noether's theorem assigns a conserved quantity to each continuous symmetry of an action. What are these quantities for the infinitely many symmetry transformations G_ε associated to RI?

i | ∢ Variation of the Lagrangian $L = -mc\sqrt{\dot{x}_\mu \dot{x}^\mu}$ under G_ε:

$$\delta_\varepsilon L = \frac{\partial L}{\partial \dot{x}^\mu}\delta_\varepsilon \dot{x}^\mu \tag{5.85a}$$

Use $\delta_\varepsilon \dot{x}^\mu := \dot{\tilde{x}}^\mu - \dot{x}^\mu = \partial_\lambda(\delta_\varepsilon x^\mu)$:

$$\stackrel{5.81}{=} -\frac{mc\dot{x}_\mu}{\sqrt{\dot{x}_\sigma \dot{x}^\sigma}}\partial_\lambda\left[-\varepsilon(\lambda)\dot{x}^\mu\right] \tag{5.85b}$$

$$= \frac{mc}{\sqrt{\dot{x}_\sigma \dot{x}^\sigma}}\left[\dot{x}_\mu \dot{\varepsilon}(\lambda)\dot{x}^\mu + \dot{x}_\mu \varepsilon(\lambda)\ddot{x}^\mu\right] \tag{5.85c}$$

$$= mc\sqrt{\dot{x}_\mu \dot{x}^\mu}\,\dot{\varepsilon}(\lambda) + mc\varepsilon(\lambda)\partial_\lambda\sqrt{\dot{x}_\mu \dot{x}^\mu} \tag{5.85d}$$

$$= \frac{\mathrm{d}}{\mathrm{d}\lambda}\underbrace{\left[mc\varepsilon(\lambda)\sqrt{\dot{x}_\mu \dot{x}^\mu}\right]}_{=:K_\varepsilon(\lambda,\dot{x}^\mu)} = \frac{\mathrm{d}K_\varepsilon}{\mathrm{d}\lambda} \tag{5.85e}$$

→ $\delta_\varepsilon L$ is a total derivative → G_ε is a continuous symmetry of S

Note that in Eq. (5.78) we assumed $\lambda_{a/b} = \varphi(\lambda_{a/b})$ which corresponds to $\varepsilon(\lambda_{a/b}) = 0 = K_\varepsilon(\lambda_{a/b}, \dot{x}^\mu)$ such the boundary terms vanish and the action is completely invariant.

ii | ↓ *Noether's (first) theorem* →

For each continuous symmetry $\delta_\varepsilon x^\mu = G_\varepsilon x^\mu$ there is a conserved Noether charge:

$$Q_\varepsilon \stackrel{*}{=} \delta_\varepsilon x^\mu \frac{\partial L}{\partial \dot{x}^\mu} - K_\varepsilon \stackrel{5.85e}{=} \varepsilon(\lambda)mc\frac{\dot{x}^\mu \dot{x}_\mu}{\sqrt{\dot{x}_\sigma \dot{x}^\sigma}} - \varepsilon(\lambda)mc\sqrt{\dot{x}_\mu \dot{x}^\mu} = 0 \tag{5.86}$$

→ The Noether charge corresponding to G_ε vanishes identically!

"Vanishing identically" means that $Q_\varepsilon(\lambda, x^\mu, \dot{x}^\mu) \equiv 0$ for *all* functions $x^\mu(\lambda)$, and not just those that satisfy the equations of motion.

- Naïvely, we expected infinitely many conserved quantities from the infinitely many symmetry generators G_n. We found them, but quite surprisingly, they turned out to be trivially zero. This is a general feature of *local* or *gauge* symmetries; here we use the reparametrization invariance of the relativistic free particle only as an example.
- So while the conserved charges of local symmetries are trivial, such symmetries have other non-trivial implications: they enforce *constraints* on the equations of motion, so that they are no longer independent. Mathematically, this is described by ↑ *Noether's* second *theorem*.

4 | We can illustrate the implications of Noether's second theorem for the relativistic free particle:

i | The Lagrangian

$$L = -mc\sqrt{\dot{x}_\mu \dot{x}^\mu} \tag{5.87}$$

leads to the conjugate momenta

$$p_\sigma = \frac{\partial L}{\partial \dot{x}^\sigma} = -\frac{mc\dot{x}_\sigma}{\sqrt{\dot{x}_\mu \dot{x}^\mu}} \tag{5.88}$$

which satisfy the identity

$$p^2 = p^\mu p_\mu = m^2 c^2 \tag{5.89}$$

- Eq. (5.89) is an *identity*, i.e., it holds for arbitrary trajectories $x^\mu(\lambda)$. In particular, $x^\mu(\lambda)$ does not need to satisfy the equations of motion for Eq. (5.89) to be valid. In Hamiltonian mechanics, such constraints are called ⁂ *primary constraints*. So our four canonical momenta p^μ are not independent!
- Eq. (5.89) is equivalent to:

$$\frac{\mathrm{d}p^2}{\mathrm{d}\lambda} = 0 \quad \Leftrightarrow \quad \left(\frac{\mathrm{d}p^\mu}{\mathrm{d}\lambda}\right) p_\mu = 0 \tag{5.90}$$

ii | ⊲ Euler-Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial L}{\partial \dot{x}^\sigma} - \frac{\partial L}{\partial x^\sigma} = \frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial L}{\partial \dot{x}^\sigma} = \frac{\mathrm{d}p_\sigma}{\mathrm{d}\lambda} = 0 \tag{5.91}$$

→ *Four* differential equations ($\sigma = 0, 1, 2, 3$) for *four* undetermined functions $x^\mu(\lambda)$.

However: Eq. (5.91) <u>not</u> independent:

$$p^\mu \frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial L}{\partial \dot{x}^\mu} = p_\mu \frac{\mathrm{d}p^\mu}{\mathrm{d}\lambda} \overset{5.90}{=} 0 \tag{5.92}$$

- Eq. (5.92) is again an *identity*, i.e., valid for *all* functions x^μ, and not only those that satisfy the equations of motion.
- As a consequence, the system of equations of motion Eq. (5.91) effectively looses one of the four equations, and is therefore *underdetermined*.

  Put differently, if you specify a spacetime position $x^\mu(\lambda = 0)$ and its first derivative $\dot{x}^\mu(\lambda = 0)$ (note that the Euler-Lagrange equations are second-order differential equations), the equations of motion do *not* determine a unique solution $x^\mu(\lambda)$. Mathematically speaking, the initial value problem is ill-posed. This is the characteristic property of a *gauge theory*.
- This makes sense in the light of reparametrization invariance: If $x^\mu(\lambda)$ solves the equations of motion, you can construct a new solution $\tilde{x}^\mu(\lambda) = x^\mu(\varphi(\lambda))$ where φ is some diffeomorphism that is the identity except for a compact subinterval somewhere in the interior of $[\lambda_a, \lambda_b]$. In particular, $\tilde{x}^\mu(\lambda) = x^\mu(\lambda)$ in the neighborhood of λ_a, such that the two solutions cannot be distinguished by their initial value and derivative. Note how important the *locality* of the symmetry is for this argument to hold!
- This is a special case of ↑ *Noether's* second *theorem* [75, 76].

iii | The fact that our theory is a gauge theory has another, at first glance surprising, consequence:

$$H = p_\mu \dot{x}^\mu - L = -\frac{mc\dot{x}_\mu \dot{x}^\mu}{\sqrt{\dot{x}_\mu \dot{x}^\mu}} + mc\sqrt{\dot{x}_\mu \dot{x}^\mu} = 0 \tag{5.93}$$

→ The (canonical) Hamiltonian vanishes identically

- ¡! This does *not* mean that there is no time-evolution in our system. The Hamiltonian Eq. (5.93) describes the "parameter evolution" in λ – which, as we have seen, can be modified arbitrarily by gauge transformations; λ has therefore *no physical significance*.

  This phenomenon will become important for the interpretation of the Einstein field equations in GENERAL RELATIVITY.

- If one fixes a gauge, the Hamiltonian that describes evolution in this parameter is non-zero in general. E.g., for the "static gauge" $\lambda = t = x^0/c$ one finds the Hamiltonian Eq. (5.49) which coincides with the relativistic energy of the particle.