Problem 7.1: Representation of the angular momentum

[Written] (5 pts.)

Learning objective

The purpose of this problem is to work out the connection between the algebra of angular momentum (see QM script chapter 4.3) and the algebra of the uncoupled two-dimensional harmonic oscillator which was discussed on problem set 6.

We start by defining the angular momentum operators

\[ L_z = \frac{\hbar}{2}(a_x^\dagger a_y - a_y^\dagger a_x) \quad L_+ = \hbar a_y^\dagger a_x \quad L_- = \hbar a_x^\dagger a_y \] (1)

in terms of creation and annihilation operators \( a_x^\dagger, a_y^\dagger \) and \( a_x, a_y \) that fulfill the commutation relations

\[ [a_\sigma, a_{\sigma'}^\dagger] = \delta_{\sigma,\sigma'} \quad \text{and} \quad [a_\sigma^\dagger, a_{\sigma'}] = 0, \quad \sigma \in \{x, y\}. \]

a) Prove that the operators in (1) satisfy the commutation relations of angular momentum:

\[ [L_z, L_\pm] = \pm \hbar L_\pm \quad [L_+, L_-] = 2\hbar L_z. \] (2)

b) Prove that

\[ L^2 = L_z^2 + \frac{1}{2} L_+ L_- + \frac{1}{2} L_- L_+ = \frac{\hbar^2}{2} N \left( \frac{N}{2} + 1 \right), \] (3)

where \( N = N_x + N_y \) with the number operators \( N_x = a_x^\dagger a_x \) and \( N_y = a_y^\dagger a_y \).

c) Show that the operators \( L_+, L_-, L_z, \) and \( L^2 \) act on the eigenstates \( |n_x, n_y\rangle \) of the operators \( N_x \) and \( N_y \) as follows:

\[ L_+ |n_x, n_y\rangle = \hbar \sqrt{n_y(n_x + 1)} |n_x + 1, n_y - 1\rangle \] (4)

\[ L_- |n_x, n_y\rangle = \hbar \sqrt{n_x(n_y + 1)} |n_x - 1, n_y + 1\rangle \] (5)

\[ L_z |n_x, n_y\rangle = \frac{\hbar}{2} (n_x - n_y) |n_x, n_y\rangle \] (6)

\[ L^2 |n_x, n_y\rangle = \frac{\hbar^2}{2} (n_x + n_y) \left( \frac{n_x + n_y}{2} + 1 \right) |n_x, n_y\rangle. \] (7)

Think of how the matrix representation of \( L_+, L_-, L_z, \) and \( L^2 \) in the basis \( |n_x, n_y\rangle \) looks like. Which states \( |n_x, n_y\rangle \) are coupled by the operators?
d) Define
\[ l = \frac{1}{2}(n_x + n_y) \quad \text{and} \quad m = \frac{1}{2}(n_x - n_y). \] (8)
and use these definitions to show that the equations (4)-(7) reduce to the familiar expressions for the \( L_+, \ L_- , \ L_z, \) and \( L^2 \) operators:
\[ L_+ |l, m\rangle = \hbar \sqrt{l(l+1) - m(m+1)} |l, m+1\rangle \] (9)
\[ L_- |l, m\rangle = \hbar \sqrt{l(l+1) - m(m-1)} |l, m-1\rangle \] (10)
\[ L_z |l, m\rangle = \hbar m |l, m\rangle \] (11)
\[ L^2 |l, m\rangle = \hbar^2 l(l+1) |l, m\rangle. \] (12)

e) Write \(|l, m\rangle\) in terms of \( a_x^\dagger, a_y^\dagger\), and the vacuum state \(|0, 0\rangle\). Interpret the obtained result.

### Problem 7.2: Symmetric top

**Problem Set Version:** 1.0 | qm2122

Consider a symmetric rigid rotor (symmetric top) where two moments of inertia are the same \( I_x = I_y \equiv I_\perp \) and the third moment of inertia has a different value \( I_z \equiv I_\parallel \).

a) Express the Hamiltonian of such a system in terms of the angular momentum operators \( L^2 \) and \( L_z \). Provide examples for molecules whose rotation can be well described by this Hamiltonian.

b) Determine the eigenvalues and eigenstates of the Hamiltonian.

### Problem 7.3: Asymmetric top

**Problem Set Version:** 1.0 | qm2122

Consider the Hamiltonian of an asymmetric top given by
\[ H = \frac{L_x^2}{2I_x} + \frac{L_y^2}{2I_y} + \frac{L_z^2}{2I_z}, \]  

where all moments of inertia, \( I_j \), are different from each other.

a) Show that the Hamiltonian commutes with \( L_z^2 \). Express the Hamiltonian in terms of \( L_z \) and the ladder operators \( L_{\pm} = L_x \pm iL_y \).

b) We split the Hilbert space \( H \) of the system into subspaces \( H'_{l} \) and \( H''_{l} \), so that the sum \( \bigoplus_{l} (H'_{l} \oplus H''_{l}) \) is the full Hilbert space again. The subspace \( H'_{l} \) contains the eigenstates \( |l, m\rangle \) of \( L_z^2 \) and \( L_z \) with \( m = l, l - 2, \ldots, -l + 2, -l \). The subspace \( H''_{l} \) contains the eigenstates \( |l, m\rangle \) with \( m = l - 1, l - 3, \ldots, -l + 3, -l + 1 \). Show that the Hamiltonian leaves the subspaces invariant, i.e. if the Hamiltonian acts on a state of a certain subspace, the resulting state belongs to the same subspace.

c) Consider the operator \( U = \exp(-i\pi L_y/\hbar) \). Make use of \( [U, H] = 0 \) and \( U |l, m\rangle = (-1)^{l-m} |l, -m\rangle \) in order to show that the full Hilbert space \( \mathcal{H} \) can be partitioned as

\[ \mathcal{H} = \bigoplus_{l} H_l \quad \text{with} \quad H_l = H'_{l_+} \oplus H'_{l_-} \oplus H''_{l_+} \oplus H''_{l_-} , \]  

where all of the subspaces are invariant under the Hamiltonian.

\textit{Hint:} At first, convince yourself that the operator \( U \) commutes with \( L_z^2 \) and that the operator \( U \) leaves the subspaces \( H'_{l} \) and \( H''_{l} \) invariant. Then, consider that the operator \( U \) couples the state \( |l, m\rangle \) to the state \( |l, -m\rangle \). The stated partition into subspaces follows from the diagonalization of the corresponding \( 2 \times 2 \) matrices.

\textit{Supplementary question:} What is the physical meaning of the operator \( U \)?

d) Determine the eigenvalues and eigenstates of the Hamiltonian for \( l = 1 \). Compare the results with problem 2.