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## Problem 6.1: Isotropic 2D Harmonic Oscillator

**[Written] (4 pts.)**

↪ ID: ex_isotropic_2d_harmonic_oscillator:qm2122

**Learning objective**

The harmonic oscillator problem is a corner stone in physics. In the lecture you solved the one-dimensional harmonic oscillator using creation and annihilation operators. In this exercise we generalize this procedure to two dimensions and you will see that the system decouples into two one-dimensional oscillators.

The Hamiltonian for an isotropic harmonic oscillator ($\omega_x = \omega_y = \omega$) in two dimensions is given by

$$H = \frac{\boldsymbol{p}^2}{2m} + \frac{m\omega^2}{2}\boldsymbol{r}^2, \qquad \text{where} \qquad \boldsymbol{p} = \begin{pmatrix} p_x \\ p_y \end{pmatrix}, \qquad \boldsymbol{r} = \begin{pmatrix} x \\ y \end{pmatrix}. \tag{1}$$

The components of $\boldsymbol{p}$ and $\boldsymbol{r}$ fulfill the canonical commutation relations,

$$[r_\alpha, p_\beta] = i\hbar\delta_{\alpha,\beta}, \qquad [r_\alpha, r_\beta] = [p_\alpha, p_\beta] = 0. \tag{2}$$

a) We introduce the operators

$$a_\alpha = \sqrt{\frac{m\omega}{2\hbar}}\left(r_\alpha + \frac{i}{m\omega}p_\alpha\right), \qquad a_\alpha^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(r_\alpha - \frac{i}{m\omega}p_\alpha\right), \tag{3}$$

with $\alpha \in \{x, y\}$. Which commutation relations do the new operators satisfy? Express the Hamiltonian in terms of $a_x^{(\dagger)}$ and $a_y^{(\dagger)}$.

*Result:*

$$H = \hbar\omega\left(a_x^\dagger a_x + \frac{1}{2}\right) + \hbar\omega\left(a_y^\dagger a_y + \frac{1}{2}\right) \tag{4}$$

b) Consider the operators $N_\alpha = a_\alpha^\dagger a_\alpha$. Since $N_\alpha^\dagger = N_\alpha$, we can introduce the orthonormal sates $|n_x, n_y\rangle$ as the eigenstates of $N_\alpha$,

$$N_x |n_x, n_y\rangle = n_x |n_x, n_y\rangle \tag{5}$$
$$N_y |n_x, n_y\rangle = n_y |n_x, n_y\rangle, \tag{6}$$

where the eigenvalues $n_\alpha \in \mathbb{R}$ are not determined yet. Determine the eigenvalues $n_\alpha$ and constuct the states $|n_x, n_y\rangle$. Start by showing

$$[N_\alpha, a_\beta] = -a_\alpha\delta_{\alpha,\beta}, \qquad \left[N_\alpha, a_\beta^\dagger\right] = a_\alpha^\dagger\delta_{\alpha,\beta}. \tag{7}$$

Then show that $a_\alpha^\dagger |n_x, n_y\rangle$ and $a_\alpha |n_x, n_y\rangle$ are still eigenstates to the operators $N_x$ and $N_y$ and normalize them. To obtain the eigenvalues $n_\alpha$, remember that the norm of a state has to be positive. Finally, construct the states $|n_x, n_y\rangle$ from the state $|0, 0\rangle$.

c) Calculate the wave function $\psi_{0,0}(\boldsymbol{r})$ of the state $|0,0\rangle$ by evaluating

$$a_x |0,0\rangle = 0 \qquad \text{and} \qquad a_y |0,0\rangle = 0 \tag{8}$$

in position space and show that it separates, $\psi_{0,0}(\boldsymbol{r}) = \phi_0(x)\phi_0(y)$, where $\phi_0$ is the ground state wave function of the one-dimensional harmonic oscillator.

d) Show that the operators $N_\alpha$ commute with the Hamiltonian $H$ and that $|n_x, n_y\rangle$ are also eigenstates to $H$. Express the eigenenergies in terms of $n_\alpha$. Are the energies degenerate? Does the measurement of the observable $N = N_x + N_y$ specify the state of the system?

## Problem 6.2: Coherent states **[Oral] (5 pts.)**

↪ ID: ex_coherent_states:qm2122

**Learning objective**

The eigenstates of the one-dimensional harmonic oscillator $|n\rangle$ are not eigenstates of the ladder operators $a^{(\dagger)}$. The coherent state which is an eigenstate of the annihilation operator $a$ is a useful object for example in quantum optics. In this exercise we investigate the properties of coherent states and see that the expectation values $\langle x\rangle$ and $\langle p\rangle$ show classical behavior.

For every $\alpha \in \mathbb{C}$, we define the coherent state

$$|\psi_\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n\geq 0} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \,, \tag{9}$$

where $|n\rangle$ is an eigenstate of the one-dimensional harmonic oscillator Hamiltonian.

a) Verify that the coherent state $|\psi_\alpha\rangle$ is an eigenstate of the annihilation operator $a$ and $\alpha$ is the eigenvalue. Show that the creation operator $a^\dagger$ has no eigenstates.

b) Show that

$$|\psi_\alpha\rangle = e^{\alpha a^\dagger - \alpha^* a} |0\rangle \,. \tag{10}$$

**Hint:** $e^{-\alpha a}|0\rangle = |0\rangle$.

c) Calculate $\langle x\rangle_\alpha = \langle\psi_\alpha|x|\psi_\alpha\rangle$, $\langle p\rangle_\alpha$, $\Delta x_\alpha$, $\Delta p_\alpha$ and show that, for all $\alpha \in \mathbb{C}$,

$$\Delta x_\alpha \Delta p_\alpha = \frac{\hbar}{2}, \tag{11}$$

which means coherent states minimize the Heisenberg uncertainty relation.

d) Show that coherent states are not orthogonal but fulfill the relation

$$\langle\psi_\alpha|\psi_\beta\rangle = e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2-2\alpha^*\beta)}. \tag{12}$$

e) Derive the time evolution of the coherent state $|\psi_\alpha(t)\rangle$ and of the expectation values $\langle x\rangle_t$ and $\langle p\rangle_t$ under the Hamiltonian $H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$.

## Problem 6.3: Angular momentum

**[Oral] (3 pts.)**

↪ ID: ex_angular_momentum:qm2122

**Learning objective**

In this exercise, we focus on the angular momentum operator. With its help we can describe rotations and introduce the concepts of scalar and vector operators.

Consider the angular momentum operator $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$. In the lecture it was shown that $\boldsymbol{L}$ is the infinitesimal generator of rotations such that rotations around some axis $\boldsymbol{n}$ with $\boldsymbol{n}^2 = 1$ about some angle $\omega$ can be written as $U_{\boldsymbol{\omega}} = \exp\left(-i\omega \boldsymbol{L} \cdot \boldsymbol{n}/\hbar\right)$. Under such rotations a *scalar* operator $S$ transforms like

$$U_{\boldsymbol{\omega}}^\dagger \, S \, U_{\boldsymbol{\omega}} = S \,, \tag{13}$$

and a *vector* operator $\boldsymbol{X}$ transforms like

$$U_{\boldsymbol{\omega}}^\dagger \, \boldsymbol{X} \, U_{\boldsymbol{\omega}} = R_{\boldsymbol{\omega}} \, \boldsymbol{X} \,, \tag{14}$$

where $R_{\boldsymbol{\omega}}$ is the usual rotation matrix in three dimensions around some axis $\boldsymbol{\omega} = \omega \boldsymbol{n}$.

a) Show that for a scalar operator $S$, $[\boldsymbol{L}, S] = 0$.
**Note:** The notation $[\boldsymbol{L}, S] = 0$ is short for $[L_i, S] = 0$, $\forall i$.

b) Using that $\boldsymbol{r}$ and $\boldsymbol{p}$ are vector operators, show that $\boldsymbol{L}$ is also a vector operator.
**Hints:**
- Consider the components of $U_{\boldsymbol{\omega}}^\dagger \, \boldsymbol{r} \times \boldsymbol{p} \, U_{\boldsymbol{\omega}}$ and show that $U_{\boldsymbol{\omega}}^\dagger \, \boldsymbol{r} \times \boldsymbol{p} \, U_{\boldsymbol{\omega}} = U_{\boldsymbol{\omega}}^\dagger \, \boldsymbol{r} \, U_{\boldsymbol{\omega}} \times U_{\boldsymbol{\omega}}^\dagger \, \boldsymbol{p} \, U_{\boldsymbol{\omega}}$.
- Use the property $R_{\boldsymbol{\omega}} \boldsymbol{a} \times R_{\boldsymbol{\omega}} \boldsymbol{b} = R_{\boldsymbol{\omega}} (\boldsymbol{a} \times \boldsymbol{b})$ of the vector product.

c) Show that $[\boldsymbol{L}, \boldsymbol{p} \cdot \boldsymbol{r}] = 0$ once by explicitely calculating the commutator and in addition by showing that $\boldsymbol{p} \cdot \boldsymbol{r}$ is a scalar operator.