Problem 3.1: Commutators

Learning objective
The commutator of operators plays a crucial role in quantum mechanics. The goal of this exercise is to learn and train to evaluate the commutator of a few basic operators.

The commutator of linear operators \( A, B \in \mathcal{L}(\mathcal{H}) \) acting on some Hilbert space \( \mathcal{H} \) is defined as

\[
[A, B] := AB - BA.
\]

(1)

a) Let \( A, B, \) and \( C \) be linear operators on \( \mathcal{H} \). Show that

\[
\]

(2a)

\[
\]

(2b)

b) Let \( x = -i\hbar \partial_x \) and \( p = -i\hbar \partial_p \) be the position and momentum operator, as introduced in the lecture. Evaluate the commutators

\[
[x, p], \quad [x, p^2], \quad [x^2, p^2], \quad [xp, p^2].
\]

(3)

c) Now let \( f \) and \( g \) be smooth functions with a Taylor series representation on \( \mathbb{R} \). Show that

\[
[p, g(x)] = -i\hbar \frac{dg(x)}{dx} \quad \text{and} \quad [x, f(p)] = i\hbar \frac{df(p)}{dp}
\]

(4)

where \( f(A) \equiv \sum_{n=0}^{\infty} f_n A^n \) with an arbitrary operator \( A \) and Taylor coefficients \( f_n \).

Problem 3.2: Baker-Campbell-Hausdorff formula

Learning objective
The Baker-Campbell-Hausdorff formula is an important relation, which is used many times in quantum mechanics in different context. In this exercise, this important formula is derived.

Consider two non-commuting operators \( A \) and \( B \), for which the relations \( [A, [A, B]] = [B, [A, B]] = 0 \) are satisfied.
a) Show that for the operators $A$ and $B$, the equality

$$e^{-At}Be^{At} = B - t [A, B]$$

is satisfied, where $t$ is an arbitrary number.

b) Show that both operators satisfy the Baker-Campbell-Hausdorff formula

$$e^{A+B} = e^A e^B e^{-\frac{[A,B]}{2}}.$$  

*Hint:* Derive a first-order differential equation for the operator $W(t) = e^{-tA}e^{t(A+B)}$ and solve it.

### Problem 3.3: Heisenberg uncertainty relations

*Oral* (2 pts.)

**Learning objective**

In this exercise, we will derive the Heisenberg uncertainty relation for the position and momentum operator. We will then, generalize this relation for two Hermitian operators.

a) Show that the momentum $p$ and position $x$ operators satisfy the uncertainty principle

$$\Delta p \cdot \Delta x \geq \frac{\hbar}{2}$$

where we have defined the standard deviation $\Delta A = \sqrt{\langle (A - \langle A \rangle)^2 \rangle}$ of an operator $A$ of expectation value $\langle A \rangle$.

*Hint:* Look up or derive your own the proof of Heisenberg uncertainty principle which is different from the proof done in lecture.

b) Given two Hermitian operators $A$ and $B$, prove the generalized uncertainty principle

$$(\Delta A)^2 \cdot (\Delta B)^2 \geq \frac{|\langle i[A,B] \rangle|^2}{4}.$$  

Show that this result is consistent with task a).