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November 2nd, 2021
WS 2021/22

Problem 3.1: Commutators **[Written] (3 pts.)**

↪ ID: ex_commutators:qm2122

> **Learning objective**
>
> The commutator of operators plays a crucial role in quantum mechanics. The goal of this exercise is to learn and train to evaluate the commutator of a few basic operators.

The commutator of linear operators $A, B \in L(\mathcal{H})$ acting on some Hilbert space $\mathcal{H}$ is defined as

$$[A, B] := AB - BA\,. \tag{1}$$

a) Let A,B, and C be linear operators on $\mathcal{H}$. Show that

$$[AB, C] = A\,[B, C] + [A, C]\,B \tag{2a}$$
$$[A, BC] = [A, B]\,C + B\,[A, C]\ . \tag{2b}$$

b) Let x and $p = -i\hbar\partial_x$ be the position and momentum operator, as introduced in the lecture. Evaluate the commutators

$$[x, p]\ ,\quad \left[x, p^2\right]\ ,\quad \left[x^2, p^2\right]\ ,\quad \left[xp, p^2\right]\ . \tag{3}$$

c) Now let f and g be smooth functions with a Taylor series representation on $\mathbb{R}$.
Show that

$$[p, g(x)] = -i\hbar\frac{dg(x)}{dx} \quad \text{and} \quad [x, f(p)] = i\hbar\frac{df(p)}{dp} \tag{4}$$

where $f(A) \equiv \sum_{n=0}^{\infty} f_n\, A^n$ with an arbitrary operator A and Taylor coefficients f_n.

Problem 3.2: Baker-Campbell-Hausdorff formula **[Written] (2 pts.)**

↪ ID: ex_Baker-Campbell-Hausdorff_formula:qm2122

> **Learning objective**
>
> The Baker-Campell-Hausdorff formula is an important relation, which is used many times in quantum mechanics in different context. In this exercise, this important formula is derived

Consider two non-commuting operators A and B, for which the relations $[A, [A, B]] = [B, [A, B]] = 0$ are satisfied.

a) Show that for the operators A and B, the equality

$$e^{-At} B e^{At} = B - t\,[A, B] \tag{5}$$

is satisfied, where t is an arbitrary number.

b) Show that both operators satisfy the Baker-Campbell-Hausdorff formula

$$e^{A+B} = e^{A} e^{B} e^{-\frac{[A,B]}{2}}. \tag{6}$$

Hint: Derive a first-order differential equation for the operator $W(t) = e^{-tA} e^{t(A+B)}$ and solve it.

Problem 3.3: Heisenberg uncertainty relations **[Oral] (2 pts.)**

↪ ID: ex_Heisenberg_uncertainty_relations:qm2122

Learning objective

In this exercise, we will derive the Heisenberg uncertainty relation for the position and momentum operator. We will then, generalize this relation for two Hermitian operators.

a) Show that the momentum p and position x operators satisfy the uncertainty principle

$$\Delta p \cdot \Delta x \geq \frac{\hbar}{2} \tag{7}$$

where we have defined the standard deviation $\Delta A = \sqrt{\langle (A - \langle A \rangle)^2 \rangle}$ of an operator A of expectation value $\langle A \rangle$.
Hint: Look up or derive your own the proof of Heisenberg uncertainty principle which is different from the proof done in lecture.

b) Given two Hermitian operators A and B, prove the generalized uncertainty principle

$$(\Delta A)^2 \cdot (\Delta B)^2 \geq \frac{|\langle \mathrm{i}\,[A, B] \rangle|^2}{4}. \tag{8}$$

Show that this result is consistent with task a).

Problem Set Version: **1.0** | **qm2122**