
Learning objective
In this problem you apply the angular momentum addition theorem. As an important use case, we consider the spin-orbit coupling in the hydrogen atom, which is the leading relativistic correction. In particular, you practice the construction of the new basis states for a fixed total angular momentum.

The spin-orbit coupling between the electron’s spin \( S \) and the orbital angular momentum \( L \) for a hydrogen atom is given by the Hamiltonian

\[
H_{LS} = f(r) L \cdot S = f(r) \sum_{\alpha=x,y,z} L_{\alpha} \otimes S_{\alpha},
\]

where \( f(r) = e^2/2m_e^2c^2r^3 \). The spin-orbit coupling can be seen as a perturbation to the non-relativistic Hamiltonian \( H_0 = p^2/2m - e^2/r \) of the hydrogen atom.

a) Define the total angular momentum operator as

\[
J = L + S = L \otimes 1 + 1 \otimes S
\]

and show that \( J^2 \) and \( J_z \) commute both with \( H_0 \) and \( H_{LS} \).

b) Consider the subspace with orbital angular momentum \( \ell \) and spin \( s \). We can write the eigenstates \( |j, m\rangle \) of \( J^2 \) and \( J_z \) as linear combinations of \( L_z \) and \( S_z \)-eigenstates \( |m_\ell, m_s\rangle \),

\[
|j, m\rangle = \sum_{m_\ell, m_s} c(m_\ell, m_s; j, m) |m_\ell, m_s\rangle.
\]

The coefficients \( c \) are called Clebsch-Gordan coefficients. Due to their ubiquity in quantum physics there are comprehensive tables available, e.g.,


Use this table to write down the change of basis (3) in the subspace with \( \ell = 1 \) and \( s = \frac{1}{2} \) explicitly.

c) The spin-orbit coupling Eq. (1) lifts the degeneracy of the \( n = 2 \) manifold of the hydrogen atom; this manifold is spanned by the two states of the \( 2s \)-orbital

\[
|n = 2, \ell = 0, m_\ell = 0; s = \frac{1}{2}, m_s \in \{-\frac{1}{2}, +\frac{1}{2}\}\rangle
\]

and the 6 states of the \( 2p \)-orbitals

\[
|n = 2, \ell = 1, m_\ell \in \{-1, 0, +1\}; s = \frac{1}{2}, m_s \in \{-\frac{1}{2}, +\frac{1}{2}\}\rangle.
\]

Use the addition of angular momentum to derive a basis \( |n, \ell, s, j, m_j\rangle \) of the \( n = 2 \) manifold such that the spin-orbit coupling term \( L \cdot S \) in Eq. (1) is diagonal.

**Hint:** Rewrite the Hamiltonian in terms of \( J^2, L^2 \) and \( S^2 \).
d) Finally, combine your results from b) and c) to determine the energy shifts due to Eq. (1) explicitly in first order perturbation theory. Show that the degeneracy of the \( n = 2 \) manifold of \( H_0 \) is lifted into three non-degenerate energies and determine the remaining degeneracy in each subspace. Explain why you have to use perturbation theory although you diagonalized \( L \cdot S \) successfully.

**Hint:** To evaluate the radial part of the matrix elements, use
\[
\langle 2, 1, m_\ell \mid \frac{1}{r^3} \mid 2, 1, m_\ell \rangle_{n=1} = \frac{1}{24a_0^3}
\]  
where \( a_0 = \hbar^2 / m_e e^2 \) is the Bohr radius.

**Problem 14.2: Identical fermions in a potential well**

[Oral] (3 pts.)

We consider two identical spin-\( \frac{1}{2} \) fermions in a one-dimensional potential given by
\[
V(x) = \begin{cases} 
0 & |x| \leq 1 \\
\infty & \text{otherwise} 
\end{cases}.
\]  
(7)

The (dimensionless) single-particle Hamiltonian for the \( i \)th particle is given by
\[
H^{(i)} = -\frac{1}{2} \frac{d^2}{dx_i^2} + V(x_i).
\]  
(8)

a) Explain why we can treat the spatial motion and the spin dynamics separately, that is, explain why we can write the single-particle eigenstates as a product of spatial- and spin wave functions. Write down the spatial wave functions and eigenenergies of the two single-particle eigenstates that are lowest in energy.

b) Determine the ground state and the ground state energy of a two-fermion system with Hamiltonian \( H = \sum_{i=1}^{2} H^{(i)} \) in the following two cases:

i. For a spin state that is antisymmetric under exchange of the two fermions, i.e., the singlet state \((|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}\).

ii. For a spin state that is symmetric under exchange of the two fermions, i.e., one of the triplet states \(|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle\) or \((|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)/\sqrt{2}\).

c) Examine the influence of a contact-interaction between the two fermions which is described by the interaction potential \( \lambda \delta(x_1 - x_2) \) with strength \( \lambda \in \mathbb{R} \). To this end, calculate the energy correction in first order perturbation theory (assuming \( |\lambda| \ll 1 \)) for both the singlet state and the triplet states.

Explain why the perturbative result for the triplet states is correct for arbitrary \( \lambda \).