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Problem 12.1: Stark Effect in the Harmonic Oscillator **[Oral] (2 pts.)**

↪ ID: ex_stark_effect_harmonic_oscillator:qm2122

Learning objective

In this exercise, we investigate how a static homogeneous electric field affects the eigenfunctions of the one-dimensional harmonic oscillator. Using perturbation theory, we calculate the corrections to the unperturbed eigenfunctions and eigenenergies and compare the results to the exact solution of the problem.

The Hamiltonian operator of a one-dimensional harmonic oscillator in a homogeneous electric field E reads

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2 + eEq. \tag{1}$$

Consider the third term in the Hamiltonian as a perturbation to the free harmonic oscillator.

a) Calculate the corrections to the eigenfunctions in first order and to the eigenenergies up to second order in the small parameter $\lambda = eE$.

b) Calculate the exact eigenvalues of the Hamiltonian (1) and compare your result with the results from (a).
Hint: Shift the position operator q by a constant factor.

Problem 12.2: Perturbation in a 2-level System **[Written] (3 pts.)**

↪ ID: ex_perturbation_2_level_system:qm2122

Learning objective

We calculate the corrections to the eigenfunctions and eigenenergies of a perturbed 2-level system. In contrast to the previous exercise, we also consider the degenerate case, where the eigenenergies of the unperturbed system coincide.

The unperturbed Hamiltonian of a 2-level system in its eigenbasis reads

$$H_0 = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}. \tag{2}$$

Consider a general perturbation of the form

$$H' = \lambda\, \boldsymbol{e} \cdot \boldsymbol{\sigma}, \tag{3}$$

with the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4}$$

and a general vector $\boldsymbol{e} = (e_x, e_y, e_z)$.

a) In the non-degenerate case ($E_1 \neq E_2$), compute the eigenfunctions in first order and the eigenenergies up to second order in $\lambda|\boldsymbol{e}| \ll 1$.

b) In the degenerate case ($E_1 = E_2$), calculate the first-order correction to the eigenenergies.

c) Find the exact eigenenergies of the perturbed Hamiltonian and compare your result with (a) and (b).

Problem 12.3: Projection Operator **[Oral] (3 pts.)**

↪ ID: ex_projection_operator:qm2122

Learning objective

With the projection operator we can project a state onto a subspace of the Hilbert space, which has many applications in quantum mechanics, e.g. it can be used to construct an appropriate basis for a given symmetry. In this exercise, we define the projection operator and show its basic properties.

A projection operator is defined by $P^2 = P$ with $P^\dagger = P$.

a) Show that its eigenvalues are 0 and 1.

b) Consider an Hilbert space spanned by $\{|\psi_i\rangle\}$ with $i \in M$ and the subspace $\{|\psi_k\rangle\}$ with $k \in K$ and $K \subseteq M$. Show that $P = \sum_{k\in K} |\psi_k\rangle \langle\psi_k|$ is a projection operator for the subspace.

c) Given an initial state $|\psi\rangle$, we perform a measurement of P and the outcome is 1. In which state is the sytem after the measurement?