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Problem 11.1: Density matrix

[Written] (4 pts.)

↪ ID: ex_density_matrix:qm2122

> **Learning objective**
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> In this problem, instead of describing a quantum state with a state vector, you will write it in the form of a *density matrix* and show that you get the same expectation values in the new formalism. You will recognize the advantage of density matrices in calculations involving mixed states which arise when we want to describe statistical ensembles of pure quantum states.

Consider a state with angular momentum $l = 1$ so that the basis is given by three states which we denote as $|+1\rangle$, $|0\rangle$, and $|-1\rangle$. Let X and Z be two observables such that

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{1}$$

and the initial state is $|\psi_0\rangle = \frac{1}{2}(|+1\rangle - \sqrt{2}|0\rangle + i|-1\rangle)$.

a) Determine the density matrix ρ_0 for the state $|\psi_0\rangle$.

b) Determine the expectation values of the observables X and Z in two ways: first using the formula with the density matrix and then using the standard expression for the state.

c) We perform a Z measurement where the measurement outcome is -1. Write down the density matrix after this measurement; does it correspond to a mixed or a pure state?

d) Now we measure the observable Z of the initial state $|\psi_0\rangle$ without recording the measurement result. Write down the new density matrix. Is it a mixed or a pure state? Determine the expectation values of the observables X and Z and explain why the expectation value of X in the new state after Z measurement is different than in the initial state $|\psi_0\rangle$.

Problem 11.2: Two-level system

[Oral] (1 pts.)

↪ ID: ex_two_level_system:qm2122

> **Learning objective**
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> In this exercise you will learn to derive the expression for the density matrix of a two-level system as a function of time from which you can deduce the system's probability to be in one of its eigenstates at any given time.

Consider the Hamiltonian

$$H = \hbar\omega \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2}$$

written in the basis $\{|a_1\rangle, |a_2\rangle\}$ where the states are labelled by the (non-degenerate) eigenvalues of an operator A.

Assume that at time $t = 0$ there is a statistical mixture described by the density matrix

$$\rho_0 = \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}. \tag{3}$$

How large is the probability $w_{a_1}(t)$ to obtain the measurement result a_1 when measuring A at a later time t ?

Problem 11.3: Harmonic oscillator in the Heisenberg picture **[Written] (5 pts.)**

↪ ID: ex_harmonic_oscillator_Heisenberg_picture:qm2122

Learning objective

In this problem you will use the Heisenberg formulation of quantum mechanics to compute the position and momentum expectation values of a one dimensional harmonic oscillator. Using the relation to the familiar operators in the Schrödinger picture (Eq. (4)), you will derive the new ladder operators ($a_H(t)$, $a_H^\dagger(t)$) and the position and momentum operators ($Q_H(t)$, $P_H(t)$) in the Heisenberg picture which will now depend on time. One goal is to study the connection between the equations of motion for operators in the Heisenberg picture and Hamilton's equations in classical mechanics.

The operators in the Heisenberg picture and the Schrödinger picture are related through

$$A_H(t) = U^{-1}(t) A_S U(t) \quad \text{with} \quad U(t) = e^{-\frac{i}{\hbar}Ht}. \tag{4}$$

$U(t)$ is the time evolution operator which obeys the Schrödinger equation $i\hbar\partial_t U(t) = HU(t)$. We define the time-independent states in the Heisenberg picture as $|\psi_H\rangle := |\psi_S(t=0)\rangle$. The index H denotes operators and states in the *Heisenberg* picture while the index S refers to the *Schrödinger* picture.

a) Derive the Heisenberg equation of motion for operators,

$$\partial_t A_H(t) = \frac{1}{i\hbar}[A_H(t), H] \tag{5}$$

starting from Eq. (4).

b) We now consider the Hamiltonian for a harmonic oscillator,

$$H = \frac{P^2}{2m} + \frac{m\omega^2}{2}Q^2 = \hbar\omega\left(a_S^\dagger a_S + \frac{1}{2}\right) \quad \text{with} \quad [a_S, a_S^\dagger] = 1. \tag{6}$$

Show that

$$a_H(t) = e^{-i\omega t} a_S \quad \text{and} \quad a_H^\dagger(t) = e^{i\omega t} a_S^\dagger. \tag{7}$$

Hint: Let $a_H(t)$ act onto an eigenstate of the harmonic oscillator or use the Baker-Campbell-Hausdorff formula.

c) Use the results of the previous task to derive relations

$$Q_H(t) = Q_H(t; P_S, Q_S) \quad \text{and} \quad P_H(t) = P_H(t; P_S, Q_S). \tag{8}$$

d) Show that for the harmonic oscillator the Heisenberg equations of motion (Eq. (5)) lead to Hamilton's classical equations of motion for the operators $Q_H(t)$ and $P_H(t)$.

e) We define at time $t = 0$ the state $|\psi_S(t = 0)\rangle = |\psi_H\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle) =: |\gamma\rangle$ (in the basis of the number operator $N|n\rangle = n|n\rangle$). Compute the expectation values

$$\langle Q_H(t)\rangle_\gamma = \langle\gamma|Q_H(t)|\gamma\rangle \quad \text{and} \quad \langle P_H(t)\rangle_\gamma = \langle\gamma|P_H(t)|\gamma\rangle \tag{9}$$

for arbitrary times t.