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Problem 10.1: Operators in position space **[Oral] (3 pts.)**

↪ ID: ex_operators_position_space:qm2122

Learning objective

In this exercise we study how operators act on states in position space.

a) Consider a self-adjoint operator $A = A^\dagger$. Show that in position space

$$\langle x|A|\psi\rangle \equiv A\psi(x) = \int \mathrm{d}y\, A(x,y)\ \psi(y) \tag{1}$$

with $A(x,y) = \langle x|A|y\rangle$.

b) Show that the matrix elements of the position operator X are given by

$$\langle x|X|y\rangle = \delta(x-y)\ x. \tag{2}$$

c) Consider the operator

$$A = \lambda - H_0 \tag{3}$$

with $\lambda \in \mathbb{C}$ and

$$H_0 = \frac{p^2}{2m}. \tag{4}$$

Let $G = A^{-1} = (\lambda - H_0)^{-1}$ be the inverse operator to A. Show that $G(x,y) = \langle x|G|y\rangle$ is the Green's function of A.

Problem 10.2: The Born series **[Written] (3 pts.)**

↪ ID: ex_born_series:qm2122

Learning objective

In this exercise, we study the Born series and derive the general expression for the expansions in real and Fourier space. Each term in the expansion can be graphically expressed with strict rules how to relate the diagram with the mathematical expression. This is an example of the famous Feynman diagrams.

a) Consider the Lippmann-Schwinger equation

$$\psi_{\boldsymbol{k}_0}(\boldsymbol{r}) = e^{i\boldsymbol{k}_0\cdot\boldsymbol{r}} + \int_{\mathbb{R}^3} \mathrm{d}^3r'\, G(\boldsymbol{r}-\boldsymbol{r}')\, V(\boldsymbol{r}')\, \psi_{\boldsymbol{k}_0}(\boldsymbol{r}')\,. \tag{5}$$

Take the Ansatz

$$\psi_{\boldsymbol{k}}(\boldsymbol{r}) = \sum_{n=0}^{\infty} \delta\psi_{\boldsymbol{k}}^{(n)}(\boldsymbol{r})\,, \qquad \delta\psi_{\boldsymbol{k}}^{(0)}(\boldsymbol{r}) = \psi_0(\boldsymbol{r}) = e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \tag{6}$$

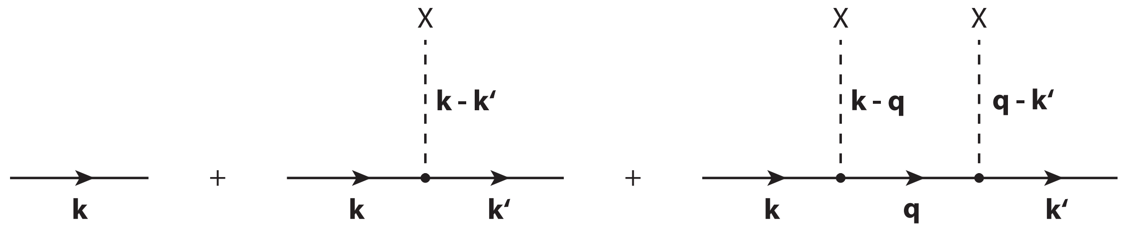
(where (n) denotes the order of the potential V) for the solution of the Lippmann-Schwinger equation. Assuming the ansatz converges, derive (e.g. by induction) the equivalent representation of the solution

$$\psi_{\boldsymbol{k}}(\boldsymbol{r}) = \psi_0(\boldsymbol{r}) + \sum_{n=1}^{\infty} X^n \psi_0(\boldsymbol{r})\,. \tag{7}$$

What is the operator X and its action on a wave function $X\psi$?

b) Find X in momentum space and compare it to its real space representation.

c) Scattering processes are typically depicted as *Feynman diagrams*. Below you can see such a diagram. Explain the meaning of each part of the diagram and give a way to translate between diagrams and the scattering series (7), the so called *Feynman rules*.



Problem 10.3: Scattering at the Yukawa potential

[Oral] (3 pts.)

↪ ID: ex_scattering_yukawa_potential:qm2122

Learning objective

The goal of this exercise is to determine the scattering amplitude within the Born approximation for the Yukawa Potential. The Yukawa potential describes the interaction between charged particles in a theory where photons have a mass μ and the Coulomb potential is recovered for vanishing photon mass $\mu \to 0$. Note that for the scattering amplitude something strange happens in this limit as the Coulomb potential is a long-range interaction.

a) The Yukawa potential is given by

$$V(r) = g\frac{\exp(-\mu r)}{r}. \tag{8}$$

Calculate the scattering amplitude in the first order Born approximation.

b) Show that for the case $\mu \to 0$ the scattering cross section of the Yukawa potential becomes the scattering cross section of Rutherford scattering

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{g^2}{8E^2 \sin^4\left(\frac{\Theta}{2}\right)}. \tag{9}$$

Here Θ denotes the deflection angle of the scattering process.

c) For a generic potential $V(\boldsymbol{r})$, derive the general expression for the second order contribution to the scattering amplitude and express it by the Fourier components of V.

*d) Determine the scattering amplitude in the second order Born approximation for the Yukawa potential.

Hint: Bring the scattering amplitude to the form

$$f^{(2)}_{\boldsymbol{k}}(\boldsymbol{k}') \propto \int d^3q \frac{1}{(\boldsymbol{q}-\boldsymbol{k})^2+\mu^2} \frac{1}{k^2-q^2+i\delta} \frac{1}{(\boldsymbol{k}'-\boldsymbol{q})^2+\mu^2}.$$

Then you can use the relation

$$\int_{-1}^{1} dz \frac{2}{[(a+b)+z(a-b)]^2} = \frac{1}{ab}$$

with $a = (\boldsymbol{q}-\boldsymbol{k})^2+\mu^2$ and $b = (\boldsymbol{q}-\boldsymbol{k}')^2+\mu^2$ to simplify the $\boldsymbol{q}$-integration.

Problem 10.4: Exact low energy scattering amplitude* **[Written] (4 pts.)**

↪ ID: ex_exact_low_energy_scattering_amplitude:qm2122

> **Learning objective**
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> The low energy scattering for all short range interactions is determined by a single parameter: the s-wave scattering length. Here we study a potential which reproduces this low energy scattering amplitude and can be solved exactly.

The scattering process at the potential V with

$$V\,\psi(\boldsymbol{r}) = \alpha(\boldsymbol{r}) \int \mathrm{d}^3 r'\, \alpha(\boldsymbol{r}')\, \psi(\boldsymbol{r}') \tag{10}$$

can be solved exactly via the Lippmann-Schwinger equation.

a) Show that the ansatz

$$\psi_{\boldsymbol{k}}(\boldsymbol{r}) = e^{i\boldsymbol{k}\cdot\boldsymbol{r}} + \lambda_{\boldsymbol{k}} \cdot \int \mathrm{d}^3 r'\, G(\boldsymbol{r}-\boldsymbol{r}')\, \alpha(\boldsymbol{r}') \tag{11}$$

with some scalar $\lambda_{\boldsymbol{k}}$ solves the Lippmann-Schwinger equation; find an expression for $\lambda_{\boldsymbol{k}}$.

What is the scattering amplitude $f_k(\Omega)$?

b) Now we choose

$$\alpha(\boldsymbol{r}) = g \cdot \frac{e^{-\kappa r}}{r} \tag{12}$$

with $r = |\boldsymbol{r}|$ and $\kappa > 0$.

In the previous subtask, the expression

$$b_{\boldsymbol{k}} = \int \mathrm{d}^3 r\, \alpha(\boldsymbol{r}) \int \mathrm{d}^3 r'\, G(\boldsymbol{r}-\boldsymbol{r}')\, \alpha(\boldsymbol{r}') \tag{13}$$

emerged.

Calculate $b_{\boldsymbol{k}}$ for our choice of $\alpha(\boldsymbol{r})$ using a mathematics program (e.g. Mathematica[1]).

Hint: Express $\alpha(\boldsymbol{r})$ in terms of the Fourier coefficients $\tilde{\alpha}(\boldsymbol{q})$ and use the Green's function in momentum space

$$\tilde{G}(\boldsymbol{q}) = \frac{1}{E - \frac{\hbar^2 q^2}{2m} + i\varepsilon} \tag{14}$$

with a small $\varepsilon > 0$.

c) Find $b_{\boldsymbol{k}}$ using the residue theorem.

d) Show that for our choice of $\alpha(\boldsymbol{r})$ and low energies ($k \to 0$), the scattering amplitude takes the form

$$f_k(\Omega) = -\frac{1}{\frac{1}{a_s} + ik}. \tag{15}$$

Calculate the scattering length a_s.

[1] A free cloud version can be found at `https://www.wolframcloud.com/`.