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Information on lecture and tutorials

Here a few infos on the modalities of the course ***"Theoretische Physik II: Quantenmechanik"***:

- The `C@MPUS-ID` of this course in 042000000.
- You can find detailed information on lecture and tutorials on the website of our institute:
  `https://www.itp3.uni-stuttgart.de/teaching/qm2122/`
- In addition, you can find information on lecture and tutorials on `ILIAS`:
  `https://ilias3.uni-stuttgart.de/goto_Uni_Stuttgart_crs_2593717.html`
- **Written** problems have to be handed in and will be corrected by the tutors. You must earn at least **80.0 %** of the written points to be admitted to the exam.
- **Oral** problems have to be prepared for the exercise session and will be presented by a student at the blackboard. You must earn at least **66.0 %** of the oral points to be admitted to the exam.
- Every student is required to **present** at least **2** of the oral problems at the blackboard to be admitted to the exam.
- Problems marked with an asterisk ($*$) are optional and can earn you bonus points.
- If you have questions regarding the problem sets, feel free to contact you tutor at any time.

Problem 1.1: Plane waves **[Written] (2 pts.)**

↪ ID: ex_plane_waves:qm2122

Learning objective

This problem serves as a repetition of basic properties of plane waves as a basis set. They will prove useful for many calculations throughout this course.

In an interval $[0, L]$ with periodic boundary conditions, plane waves are given by

$$\psi_n = \frac{1}{\sqrt{L}} \exp\left(\frac{i}{\hbar} p_n x\right), \quad \text{where} \quad p_n = \frac{2\pi\hbar n}{L} \quad \text{and} \quad n \in \mathbb{Z}. \tag{1}$$

a) Show that they form a complete set of orthonormal basis functions. In other words, show that:

$$\int_0^L dx\, \psi_n^*(x)\psi_m(x) = \delta_{n,m} \qquad \text{(orthonormality)}\,, \tag{2}$$

$$\sum_{n=-\infty}^{\infty} \psi_n^*(x)\psi_n(x') = \delta(x-x') \qquad \text{(completeness)}\,. \tag{3}$$

b) In the limit $L \to \infty$, the orthogonality and completeness relations read

$$\int_{-\infty}^{\infty} dx\, \exp\left[-\frac{i}{\hbar}(p-p')x\right] = 2\pi\hbar\,\delta(p-p') \quad \text{(orthogonality)}\,, \tag{4}$$

$$\int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar}\, \exp\left[\frac{i}{\hbar}(x-x')p\right] = \delta(x-x') \quad \text{(completeness)}\,. \tag{5}$$

Perform the limit $L \to \infty$ explicitly for ψ_n by using an appropriate prefactor and show both the orthogonality (4) and the completeness (5). (Note that the basis functions are no longer normalizable and are thus called a *general set of basis functions*.)

Problem 1.2: Fourier transform **[Written] (6 pts.)**

↪ ID: ex_fourier_transform:qm2122

Learning objective

In this problem, which serves as a repetition, you will prove important properties of the Fourier transform.

Given a complex-valued function $f \in L^{(1)} = \{f : \mathbb{R} \to \mathbb{C} \mid \int_{\mathbb{R}} |f|dx < \infty\}$, the one-dimensional Fourier transform is defined as

$$\mathcal{F}[f(x)] = \int_{-\infty}^{\infty} \mathrm{d}x\, e^{-ikx} f(x) \equiv \hat{f}(k)\,. \tag{6}$$

a) Show that

$$\mathcal{F}[1] = 2\pi\delta(k) \quad \text{and} \quad \mathcal{F}[\delta(x)] = 1\,, \tag{7}$$

where $\delta(q)$ is the Dirac delta function. The Fourier transformation of a distribution is defined by its action on a test function, i.e., use $\int_{-\infty}^{\infty} \mathrm{d}q\, h(q)\delta(q) = h(0)$.

b) Show that the inverse Fourier transform is

$$\mathcal{F}^{-1}[\hat{f}(k)] = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi}\, e^{ikx} \hat{f}(k) = f(x)\,. \tag{8}$$

c) Show that

$$\mathcal{F}[f(x+a)] = e^{ika}\, \mathcal{F}[f(x)]\,. \tag{9}$$

d) Show that

$$\mathcal{F}[\partial_x f(x)] = ik\, \mathcal{F}[f(x)] \; . \tag{10}$$

e) The convolution of two functions $f : \mathbb{R} \to \mathbb{C}$ and $g : \mathbb{R} \to \mathbb{C}$ is defined as

$$(f * g)(x) = \int_{-\infty}^{\infty} \mathrm{d}\xi\; f(\xi) g(x - \xi) = \int_{-\infty}^{\infty} \mathrm{d}\xi\; f(x - \xi) g(\xi) \; . \tag{11}$$

Show that

$$\mathcal{F}[(f * g)(x)] = \mathcal{F}[f(x)] \cdot \mathcal{F}[g(x)] \; . \tag{12}$$

f) Calculate the Fourier transform of the Gaussian function

$$f(x) = e^{-ax^2} \quad \text{with} \quad a \neq 0 \;, \; \mathrm{Re}(a) \geq 0 \;, \; a \in \mathbb{C} \; . \tag{13}$$

Pay particular attention to the calculation for a purely imaginary a.

Hint: Determine $\partial_k \mathcal{F}[f](k)$ and use $\int_{-\infty}^{\infty} e^{-ax^2} = \sqrt{\pi/a}$.

Problem 1.3: Classical action **[Oral] (2 pts.)**

↪ ID: ex_classical_action:qm2122

Learning objective

This problem deals with the classical action and Lagrangian which serves both as a repetition of classical mechanics as well as a starting point for deriving dynamical equations in quantum mechanics.

In classical mechanics, the action S is defined as

$$S[q](t) = \int_{t_0}^{t} dt'\, L(\dot{q}(t'), q(t'), t') \,, \tag{14}$$

where L is the *Lagrangian function* (also called *Lagrangian*) and q is some generalized coordinate of the system.

a) Calculate the action $S(q_\tau, \tau)$ along classical paths for the following systems:

- free particle,
- harmonic oscillator with $V(q) = \frac{1}{2} m\omega^2 q^2$,
- constant force F,

where $q(t = \tau_0) = 0$ is the starting point and $q(t = \tau) = q_\tau$ is the endpoint of the path at some time $\tau > \tau_0$.

b) For classical paths with a fixed starting point $q(\tau_0) = q_0$, one can interpret S as a function of q_τ and τ, i.e. $S = S(q_\tau, \tau)$. Show that

$$\frac{\partial S}{\partial q_\tau} = p_\tau \,, \quad \frac{\partial S}{\partial \tau} = -H \,, \tag{15}$$

where H is the Hamiltonian function of the system.

Hint: For some fixed starting point q_0, each path can be parametrized by some arbitrary point q_τ on that path such that $q(t)$ itself can be thought of as a function of both q_τ and some general time t.