

**Problem 6.1: Feynman diagrams for  $\phi^4$ -theory**

[Written | 4 (+1 bonus) pt(s)]

ID: ex\_feynman\_diagrams\_phi4\_theory:qft26

**Learning objective**

The purpose of this problem is to become familiar with Feynman diagrams and their corresponding perturbative expressions. To this end, we use the interacting  $\phi^4$ -theory and focus on its four-point correlator to apply the machinery of real- and momentum-space Feynman diagrams.

We consider the  $\phi^4$ -theory

$$H = \frac{1}{2} \int d^3x \left[ \pi^2(\mathbf{x}) + (\nabla\phi(\mathbf{x}))^2 + m^2\phi^2(\mathbf{x}) + 2 \frac{\lambda}{4!} \phi^4(\mathbf{x}) \right] \quad (1)$$

with interacting fields  $\phi(x) = e^{iHt}\phi(\mathbf{x})e^{-iHt}$  and vacuum  $|\Omega\rangle$ .

- a) Draw all *relevant* Feynman diagrams (i.e., without vacuum bubbles) for the perturbative expansion of the four-point function 1pt(s)

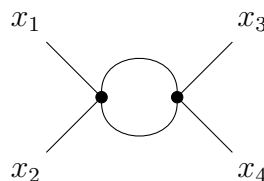
$$\langle \Omega | \mathcal{T} \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) | \Omega \rangle \quad (2)$$

up to second order ( $\lambda^2$ ).

Draw two relevant diagrams of third order ( $\lambda^3$ ): one connected and one disconnected.

**Hint:** Ignore symmetry factors and permutations of external points. Use that four-point diagrams are either fully connected or decompose into products of disjoint two-point diagrams. Up to permutations, there are 3 connected diagrams and 6 additional disconnected diagrams up to second order.

- \*b) Draw all diagrams of third order. How many are connected and disconnected, respectively (again up to permutations)? +1pt(s)
- c) Using the *real-space Feynman rules*, write down the term described by the Feynman diagram 1pt(s)



- d) Label the Feynman diagram above with directed momenta and write down the corresponding expression as prescribed by the *momentum-space Feynman rules*. 1pt(s)
- e) Use the Fourier expansion of the Feynman propagator 1pt(s)

$$D_F(x - y) = \int \frac{d^4p}{(2\pi)^4} \frac{i e^{-ip \cdot (x-y)}}{p^2 - m^2 + i\epsilon} \quad (3)$$

to show that the expressions of c) and d) are equivalent.

**Problem 6.2: Feynman rules for the interacting complex Klein-Gordon field** [ Oral | 4 pt(s) ]

ID: ex\_feynman\_rules\_interacting\_complex\_klein\_gordon\_field:qft26

**Learning objective**

Here you derive the Feynman rules for the complex Klein-Gordon field with an arbitrary interaction potential. Generically, this interaction violates causality and the resulting theory is no longer a relativistic quantum field theory. However, in condensed matter physics such theories can be used to describe the low-energy physics of interacting models that are otherwise hard to tackle analytically. This demonstrates that diagrammatic methods for perturbation theory are not restricted to relativistic high-energy physics.

Recall the (free) complex Klein-Gordon field (Problem 2.2) with Hamiltonian

$$H_0 = \int d^3x (\pi^\dagger \pi + \nabla \phi^\dagger \nabla \phi + m^2 \phi^\dagger \phi) \tag{4}$$

and fields that satisfy the canonical commutation relations  $[\phi(\mathbf{x}), \pi(\mathbf{y})] = i\delta^{(3)}(\mathbf{x} - \mathbf{y})$ .

Let  $V : \mathbb{R}^3 \rightarrow \mathbb{R}$  be a symmetric  $[V(\mathbf{r}) = V(-\mathbf{r})]$  but otherwise arbitrary (well-behaved) potential. Here we consider the interacting theory

$$H = H_0 + \frac{\lambda}{2} \int d^3x \int d^3y V(\mathbf{x} - \mathbf{y}) \phi^\dagger(\mathbf{x}) \phi^\dagger(\mathbf{y}) \phi(\mathbf{x}) \phi(\mathbf{y}) \tag{5}$$

with small parameter  $\lambda$ .

At an arbitrary time  $t_0$ , we can expand the interacting field  $\phi(t_0, \mathbf{x})$  into modes,

$$\phi(t_0, \mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_{\mathbf{p}} e^{i\mathbf{p}\mathbf{x}} + b_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\mathbf{x}}), \tag{6}$$

with the mode algebra

$$[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) \quad \text{and} \quad [b_{\mathbf{p}}, b_{\mathbf{q}}^\dagger] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) \tag{7}$$

(all other commutators vanish). In the interaction picture, we then have

$$\phi_I(x) = e^{iH_0(t-t_0)} \phi(t_0, \mathbf{x}) e^{-iH_0(t-t_0)} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_{\mathbf{p}} e^{-ipx} + b_{\mathbf{p}}^\dagger e^{ipx}) \tag{8}$$

with  $x^0 = t - t_0$ . Note that this is just the time evolution of the free theory  $H_0$  that you derived in Problem 2.2 b).

- a) Let the contraction be defined as difference between time ordering and normal ordering: 1pt(s)

$$\overline{AB} \equiv \mathcal{T}\{AB\} - :AB: \tag{9}$$

where  $A, B \in \{\phi_I, \phi_I^\dagger\}$ .

Use the decomposition  $\phi_I = \phi_a^+ + \phi_b^-$  and  $\phi_I^\dagger = \phi_a^- + \phi_b^+$  into positive- and negative-frequency parts (and your knowledge from the real Klein-Gordon field) to show that

$$\overline{\phi_I(x) \phi_I(y)} = \overline{\phi_I^\dagger(x) \phi_I^\dagger(y)} = 0 \tag{10a}$$

$$\overline{\phi_I^\dagger(x) \phi_I(y)} = \overline{\phi_I(x) \phi_I^\dagger(y)} = D_F(x - y) = \int \frac{d^4p}{(2\pi)^4} \frac{i e^{-ip \cdot (x-y)}}{p^2 - m^2 + i\epsilon}. \tag{10b}$$

b) Prove Wick's theorem for the free complex scalar field. That is, show that 1pt(s)

$$\mathcal{T}\{ABC \dots\} = :ABC \dots: + \{\text{all contractions between pairs of } \phi \text{ and } \phi^\dagger\} \quad (11)$$

for  $A, B, C, \dots \in \{\phi_I, \phi_I^\dagger\}$ .

**Hint:** Use induction (as in Peskin & Schroeder) with the decomposition of  $\phi$  and  $\phi^\dagger$  from above.

c) As shown in the lecture (or in Problem 5.1), time-ordered correlation functions can be rewritten in terms of interaction picture fields via 1pt(s)

$$\langle \Omega | \mathcal{T}\{ABC \dots\} | \Omega \rangle = \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{\langle 0 | \mathcal{T}\{A_I B_I C_I \dots \exp\left(-i \int_{-T}^T dt H_I(t)\right)\} | 0 \rangle}{\langle 0 | \mathcal{T} \exp\left(-i \int_{-T}^T dt H_I(t)\right) | 0 \rangle} \quad (12)$$

for  $A, B, C, \dots \in \{\phi, \phi^\dagger\}$ . Here  $|\Omega\rangle$  is the interacting vacuum and the interaction picture Hamiltonian is given by

$$H_I(t) = \frac{\lambda}{2} \int d^3x \int d^3y V(\mathbf{x} - \mathbf{y}) \phi_I^\dagger(x) \phi_I^\dagger(y) \phi_I(x) \phi_I(y). \quad (13)$$

Use this prescription in combination with Wick's theorem to evaluate the two-point correlator

$$\langle \Omega | \mathcal{T} \phi(x) \phi^\dagger(y) | \Omega \rangle \quad (14)$$

up to first order in  $\lambda$ .

Compare your result to the  $\phi^4$ -theory.

d) Use the dictionary 1pt(s)

$$y \text{ ---} \text{---} x = \overline{\phi_I(x) \phi_I^\dagger(y)} = D_F(x - y) \quad (15a)$$

$$u \text{ - - - - - } w = V(\mathbf{u} - \mathbf{w}) \delta(u^0 - w^0) \quad (15b)$$

to recast the summands found in c) as Feynman diagrams.

Generalize your result to the Feynman rules of the interacting theory of a complex scalar field with interaction potential  $V$ .