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Problem 11.1: Dimensional Regularization

[Written | 5 pt(s)]

ID: ex_dimensional_regularization:qft26

Learning objective

In this exercise we will work on the technical details of *dimensional regularization* (due to 't Hooft and Veltman). Dimensional regularization preserves the symmetries of QED and a broader class of more general theories. The idea of dimensional regularization is to extend the definition of d -dimensional volume integrals to arbitrary $d \in \mathbb{R}$. If the divergences of integrals from Feynman diagrams vanish for $d < 4$, they can be regularized if the limit $d \rightarrow 4$ is taken after evaluating physical quantities.

Let us consider spacetime to have one time dimension and $(d - 1)$ space dimensions ($d = 2, 3, 4, \dots$).

We are interested in solving integrals of the form

$$\int \frac{d^d \ell_E}{(2\pi)^d} \frac{1}{(\ell_E^2 + \Delta)^2} = \int \frac{d\Omega_d}{(2\pi)^d} \int d\ell \frac{\ell^{d-1}}{(\ell^2 + \Delta)^2} \quad (1)$$

where we have Wick-rotated the time dimension so that $d^d \ell_E$ is the volume element of d -dimensional *Euclidean* space; $d\Omega_d$ denotes the angular part of the integral in d -dimensional spherical coordinates.

a) The first factor in Eq. (1) contains the area of a unit sphere in d dimensions. Show that

1pt(s)

$$\int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \quad (2)$$

Use $\int dx e^{-x^2} = \sqrt{\pi}$ and the definition of the Gamma function $\Gamma(t) := \int_0^\infty dx x^{t-1} e^{-x}$.

b) With the result from a), show that Eq. (1) evaluates to

1pt(s)

$$\int \frac{d^d \ell_E}{(2\pi)^d} \frac{1}{(\ell_E^2 + \Delta)^2} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(2 - \frac{d}{2})}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{2 - \frac{d}{2}}. \quad (3)$$

To this end, use the substitution $x = \Delta/(\ell^2 + \Delta)$ and the definition of the beta function

$$B(\alpha, \beta) := \int_0^1 dx x^{\alpha-1} (1-x)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \quad (4)$$

The expression (3) can now be used to *define* the left-hand side for $d \in \mathbb{R}$.

Where are the poles of this generalized integral in d “dimensions”?

- c) Define $\epsilon = 4 - d$ and use the infinite product representation

1pt(s)

$$\frac{1}{\Gamma(x)} = x e^{\gamma x} \prod_{n=1}^{\infty} \left(1 + \frac{x}{n}\right) e^{-x/n} \quad (5)$$

(γ is the Euler-Mascheroni constant) to expand $\Gamma(2 - \frac{d}{2})$ to first order in ϵ .

- d) Show that the integral (3) takes the asymptotic form

1pt(s)

$$\int \frac{d^d \ell_E}{(2\pi)^d} \frac{1}{(\ell_E^2 + \Delta)^2} \xrightarrow{d \rightarrow 4} \frac{1}{(4\pi)^2} \left[\frac{2}{\epsilon} + \log \frac{4\pi}{\Delta} - \gamma + \mathcal{O}(\epsilon) \right]. \quad (6)$$

This expression extracts the diverging part of the integral for $d \rightarrow 4$ and allows for the controlled treatment of such integrals.

- e) Following the steps in part b), verify the more general expressions

1pt(s)

$$\int \frac{d^d \ell_E}{(2\pi)^d} \frac{1}{(\ell_E^2 + \Delta)^n} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(n - \frac{d}{2})}{\Gamma(n)} \left(\frac{1}{\Delta} \right)^{n - \frac{d}{2}}, \quad (7a)$$

$$\int \frac{d^d \ell_E}{(2\pi)^d} \frac{\ell_E^2}{(\ell_E^2 + \Delta)^n} = \frac{1}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(n - \frac{d}{2} - 1)}{\Gamma(n)} \left(\frac{1}{\Delta} \right)^{n - \frac{d}{2} - 1}. \quad (7b)$$

These integrals are useful for the renormalization of the electric charge (see lecture).

Problem 11.2: Thomas-Fermi screening

[Oral | 4 pt(s)]

ID: ex_thomas_fermi_screening:qft26

Learning objective

As already demonstrated in previous tasks, the machinery of quantum field theory is not restricted to high-energy physics and fundamental theories like QED; its application to condensed matter physics provides one of the most powerful tools to study strongly correlated quantum matter. In this exercise, we will study the so called *Thomas-Fermi screening* of electrons in a degenerate electron gas of density n at zero temperature.

- a) Similar to the lecture, define $\Pi(q)$ to be the sum of all *one-particle-irreducible* diagrams contributing to the photon self-energy. Show by diagrammatically expanding the *full* photon propagator $D_{\text{ph}}(q)$ that

1pt(s)

$$D_{\text{ph}}(q) = \frac{D_{\text{ph}}^0(q)}{1 - D_{\text{ph}}^0(q)\Pi(q)}, \quad (8)$$

where $D_{\text{ph}}^0(q)$ is the bare photon propagator.

This approach is related to the so called *Lindhard theory* in condensed matter theory used for calculating the effects of electric field screening by electrons.

- b) In condensed matter theory, the bare photon propagator in momentum space is simply given by the Fourier transform $U(\mathbf{q})$ of the (time-independent) interaction potential. Then, the denominator in (8) can be seen as a dielectric function given (in the static limit) as 1pt(s)

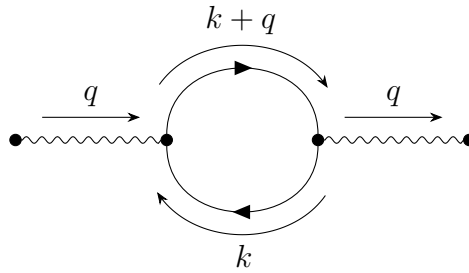
$$\varepsilon(\mathbf{q}) = 1 - U(\mathbf{q})\Pi(\mathbf{q}). \quad (9)$$

Show that the bare Coulomb interaction in momentum space, $U(\mathbf{q}) = e^2/q^2$, is now modified to an effective interaction due to the screening of the electron gas in the long wavelength limit (i.e., Π is evaluated at $\mathbf{q} = \mathbf{0}$):

$$U_{\text{eff}}(\mathbf{q}) = \frac{e^2}{q^2 + \lambda_{\text{TF}}^{-2}}, \quad (10)$$

where λ_{TF}^{-1} is the *Thomas-Fermi wave vector*.

- c) Calculate the Fourier transform $U_{\text{eff}}(\mathbf{x})$ of the effective potential (10) and discuss your result. 1pt(s)
 d) Calculate the Thomas-Fermi wave vector in the long wavelength limit ($\mathbf{q} \rightarrow \mathbf{0}$) and in the so-called *random-phase approximation*, where $\Pi(\mathbf{q})$ consists only of the particle-hole(=antiparticle) loop (neglecting the in- and outgoing lines): 1pt(s)



According to the Feynman rules in condensed matter theory, $\Pi(\mathbf{q})$ is given by

$$\Pi(\mathbf{q}) = -2i \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi} G^0(\omega, \mathbf{k}) G^0(\omega, \mathbf{k} + \mathbf{q}), \quad (11)$$

where the propagator/Green's function reads

$$G^0(\omega, \mathbf{k}) = \frac{1}{\omega - \xi(\mathbf{k}) + i\delta \text{sign}(\xi(\mathbf{k}))} \quad (12)$$

with $\xi(\mathbf{k}) = \frac{k^2}{2m} - E_F$ and E_F the Fermi energy. δ is to be taken positive but small (i.e. $\delta \rightarrow 0^+$) and $\text{sign}(x)$ refers to the signum function, which gives the sign of x and $\text{sign}(0) = 0$.

Hint: In 3D, the Fermi energy is given by $E_F = (3\pi^2 n)^{2/3} / (2m)$ with electron density n and mass m .