Prof. Dr. Mathias Scheurer
Institute for Theoretical Physics III, University of Stuttgart

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Problem 9.1: Gaussian disorder

[**Oral** | 4 (+2 bonus) pt(s)]

ID: ex_gaussian_impurity:qft25

Learning objective

In this exercise, we will once again go through the replica trick introduced in the lecture and apply it to the calculation of the quasiparticle's lifetime in the presence of Gaussian disorder. We will see how disorder-induced scattering affects the imaginary part of the self-energy, which directly determines the quasiparticle lifetime. As we shall see, this has direct physical significance in transport phenomena—it appears as the collision time $\tau_{\boldsymbol{p}}$ in the kinetic equation and ultimately determines the conductivity in the Drude formula.

Let us start by splitting the action into two parts,

$$S = S_c[\psi^\dagger, \psi] + S_d[\psi^\dagger, \psi, V],$$

where the S_c describes the clean electron system, and S_d corresponds to the coupling between electrons and the disorder potential,

$$S_d[\psi^\dagger, \psi, V] = \sum_{\omega_n} \int d^d r \psi^\dagger(\omega_n, \boldsymbol{r}) V(\boldsymbol{r}) \psi(\omega_n, \boldsymbol{r}).$$

We will consider the disorder potential to be Gaussian, namely,

$$\langle V(\boldsymbol{r}) V(\boldsymbol{r}') \rangle_V = \frac{\gamma^2}{(2\pi s^2)^{d/2}} e^{-|\boldsymbol{r}-\boldsymbol{r}'|^2/(2s^2)}, \tag{1}$$

where γ represents the strength of the disorder, and s how quickly the disorder correlation decays with distance. The averaging over the disorder $\langle . \rangle_V$ is defined as

$$\langle \mathcal{A}[V] \rangle_V = \mathcal{N} \int D[V] \mathcal{A}[V] \exp\left[-\frac{1}{L^{2d}} \frac{(2\pi s^2)^{d/2}}{2\gamma^2} \int d^d r \int d^d r' V(\boldsymbol{r}) e^{|\boldsymbol{r}-\boldsymbol{r}'|^2/(2s^2)} V(\boldsymbol{r}') \right],$$

where $\mathcal{N}$ is a normalization constant. Then the average value of a fermionic operator $\mathcal{O}$ reads

$$\langle \langle \mathcal{O} \rangle_S \rangle_V = \int D[V] P[V] \int D[\psi^\dagger, \psi] \mathcal{O}[\psi^\dagger, \psi] e^{-S_c[\psi^\dagger,\psi] - S_d[\psi^\dagger,\psi,V]},$$

$$P[V] = \mathcal{N} \exp\left[-\frac{1}{L^{2d}} \frac{(2\pi s^2)^{d/2}}{2\gamma^2} \int d^d r \int d^d r' V(\boldsymbol{r}) e^{|\boldsymbol{r}-\boldsymbol{r}'|^2/(2s^2)} V(\boldsymbol{r}') \right]$$

Even though we have already set up everything to use the perturbation theory straightforwardly, let us proceed with the *replica trick*. The basic idea behind the replica trick is to introduce R identical copies of the considered system, average over all the copies and finally take the limit $R \to 0$,

$$\langle \langle \mathcal{O} \rangle_S \rangle_V = \lim_{R\to 0} \frac{1}{R} \int D[V] P[V] \int D[\psi^\dagger, \psi] \sum_{a=1}^{R} \mathcal{O}[\psi_a^\dagger, \psi_a] e^{-\sum_{a=1}^{R} S[\psi_a^\dagger, \psi_a, V]}. \tag{2}$$

a) Perform the integration over the disordered potential $V(\boldsymbol{r})$ with the correlator given by (1) and find the part of the new action corresponding to the effective electron-electron interaction **1**pt(s)

$$\langle\langle \mathcal{O}\rangle_S\rangle_V = \lim_{R\to 0}\frac{1}{R}\int D[\psi_a,\psi_a^\dagger]\sum_{a=1}^{R}\mathcal{O}[\psi_a^\dagger,\psi_a]e^{-\sum_{a=1}^{R}S_c[\psi_a^\dagger,\psi_a]}e^{-\sum_{a,b=1}^{R}\Delta S[\psi_a^\dagger,\psi_a;\psi_b^\dagger,\psi_b]},$$

where ΔS is given in real space by

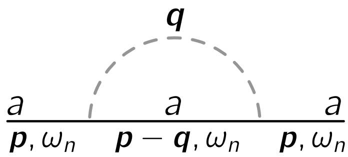
$$\Delta S[\psi_a^\dagger,\psi_a;\psi_b^\dagger,\psi_b] = -\frac{\gamma^2}{2(2\pi s^2)^{d/2}}\sum_{\omega_n,\omega_{n'},a,b}\int d^dr\int d^dr' \times \psi_a^\dagger(\omega_n,\boldsymbol{r})\psi_a(\omega_n,\boldsymbol{r})e^{-|\boldsymbol{r}-\boldsymbol{r}'|^2/(2s^2)}\psi_b^\dagger(\omega_{n'},\boldsymbol{r}')\psi_b(\omega_{n'},\boldsymbol{r}') \tag{3}$$

Also write the momentum representation of ΔS.

b) Let us now focus on calculating the perturbed Green's function **2**pt(s)

$$\tilde{G}(\omega_n,\boldsymbol{p}) = \lim_{R\to 0}\frac{1}{R}\sum_{a=1}^{R}\langle\langle\psi_a(\omega_n,\boldsymbol{p})\psi_a^\dagger(\omega_n,\boldsymbol{p})\rangle_S\rangle_V\,.$$

Recall the explanation about why only the following diagram corresponds to the lowest-order (with respect to the disorder-induced interaction) correction in the replica limit.



Identify the corresponding self-energy $\Sigma(\boldsymbol{p},\omega_n)$ and calculate its imaginary part to find the lifetime of the quasiparticles near the Fermi surface. For convenience, consider only the 3D case.

Hint: You will need to perform the analytic continuation $\Sigma^R(\omega,\boldsymbol{p}) = \Sigma(\omega_n,\boldsymbol{p})|_{i\omega_n\to\omega+i0^+}$, and then use the "on-shell" approximation assuming that one can replace ω with the particle energy $\xi_{\boldsymbol{p}}$ to find the quasiparticle lifetime $\tau_{\boldsymbol{p}}^{-1} = \mathrm{Im}\Sigma^R(\xi_{\boldsymbol{p}},\boldsymbol{p})$.

c) For the particles on the Fermi-surface consider two limits, $sp_{\mathrm{F}} \ll 1$ and $sp_{\mathrm{F}} \gg 1$ (p_{F} is the momentum at Fermi surface) and simplify the obtained results. **1**pt(s)

*d) In the quasiclassical limit, when $p_{\mathrm{F}}l \ll 1$, where l is the mean free path of electrons, the electron gas can be described by the distribution function $f_{\boldsymbol{p}}(\boldsymbol{r},t)$ – one can think of it as the number of electrons with momentum $\boldsymbol{p}$ at time t in the vicinity of point $\boldsymbol{r}$. It satisfies the following so-called *Kinetic equation* **+2**pt(s)

$$\frac{df_{\boldsymbol{p}}(\boldsymbol{r},t)}{dt} = \left[\frac{\partial}{\partial t} + \boldsymbol{F}(\boldsymbol{r},t)\cdot\frac{\partial}{\partial\boldsymbol{p}} + \boldsymbol{v}_{\boldsymbol{p}}\cdot\frac{\partial}{\partial\boldsymbol{r}}\right]f_{\boldsymbol{p}} = I_{\mathrm{col}}[f_{\boldsymbol{p}}(\boldsymbol{r},t)],$$

where $\boldsymbol{F}$ describes an external generalised force applied to the system, and $I_{\mathrm{col}}[f_{\boldsymbol{p}}(\boldsymbol{r},t)]$ is the collision integral describing (as suggested by its name) different collision, e.g. scattering on impurities, phonons, and electron-electron scattering processes. Using the distribution function one can find different macroscopic observables, for instance, the electron and current densities are given by

$$n(\boldsymbol{r},t) = \frac{1}{L^d}\sum_{\boldsymbol{p}}f_{\boldsymbol{p}}(\boldsymbol{r},t), \quad \text{and} \quad \boldsymbol{j}(\boldsymbol{r},t) = \frac{e}{L^d}\sum_{\boldsymbol{p}}\boldsymbol{v}_{\boldsymbol{p}}f_{\boldsymbol{p}}(\boldsymbol{r},t).$$

Suppose we want to find the current induced by the external homogeneous electric field $\boldsymbol{F}(\boldsymbol{r}, t) = \boldsymbol{E}$. First, linearize the kinetic equation with respect to the electric field, assuming that $f_{\boldsymbol{p}}(\boldsymbol{r}, t) \approx f_0(\varepsilon_{\boldsymbol{p}}) + \delta f_{\boldsymbol{p}}$ with $f_0(\varepsilon_{\boldsymbol{p}}) = \left(1 + e^{\beta(\varepsilon_{\boldsymbol{p}} - \mu)}\right)^{-1}$, $\delta f_{\boldsymbol{p}} \sim E$, and $I_{\mathrm{col}}[f_0] = 0$, $I_{\mathrm{col}}[\delta f_{\boldsymbol{p}}] = -\delta f_{\boldsymbol{p}}/\tau_{\boldsymbol{p}}$, where $\tau_{\boldsymbol{p}}$ is the life-time found earlier. *Try to come up with an intuitive explanation for why the time evolution of the distribution function, caused by the collision with impurities, should be related to the lifetime.* Then calculate the current density to obtain the *Drude* formula.