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Problem 8.1: Electromagnetic response of superconductors

[**Oral** | 8 pt(s)]

ID: ex_em_super1:qft25

Learning objective

So far we have primarily focused on the mechanisms leading to the superconducting phase. In this exercise we will investigate the relevant terms in the effective action of a superconductor, when coupled to an electromagnetic field.

The action of a non-interacting system, coupled to a vector potential $\boldsymbol{A}$ and scalar potential ϕ is given by

$$S_0 = \int d^d r \int d\tau \psi^\dagger{}_\sigma \left(\partial_\tau + i\phi + \frac{1}{2m}(-i\boldsymbol{\nabla} - \boldsymbol{A})^2 - \mu \right) \psi_\sigma. \tag{1}$$

This is known as minimal-coupling of an electromagnetic field. We work in units in which the electron charge is set to $e = 1$. The full action of a superconductor can then be written as

$$S = \int d\tau d^d r \left(\psi^\dagger{}_\sigma \left(\partial_\tau + i\phi + \frac{1}{2m}(-i\nabla - \mathbf{A})^2 - \mu \right) \psi_\sigma - V \psi^\dagger{}_\uparrow \psi^\dagger{}_\downarrow \psi_\downarrow \psi_\uparrow \right). \tag{2}$$

Under gauge transformations the fields behave as $\psi \to e^{i\theta}\psi$, $\psi^\dagger \to e^{-i\theta}\psi^\dagger$, $\phi \to \phi - \partial_\tau \theta$ and $\boldsymbol{A} \to \boldsymbol{A} + \boldsymbol{\nabla}\theta$.

a) Perform a Hubbard–Stratonovich transformation in the Cooper channel and show that **1** pt(s)

$$S = \int d^d r \int d\tau \left[\psi^\dagger{}_\sigma \left(\partial_\tau + i\phi + \frac{1}{2m}(-i\boldsymbol{\nabla} - \boldsymbol{A})^2 - \mu \right) \psi_\sigma + \left(\frac{1}{V}|\Delta|^2 + \left(\Delta \psi^\dagger{}_\uparrow \psi^\dagger{}_\downarrow + \bar{\Delta} \psi_\downarrow \psi_\uparrow \right) \right) \right]. \tag{3}$$

Are Δ and $\bar{\Delta}$ observables? Justify your answer by analyzing how they transform under gauge transformations.

b) Now, we fix $\Delta_0 = |\Delta|$, and integrate out the fermions ψ, to obtain the effective action as, **2** pt(s)

$$S_{EM}[\theta, \boldsymbol{A}, \phi] := -\mathrm{Tr} \ln \left(\mathcal{G}^{-1} \left[\Delta = \Delta_0 e^{2i\theta}, \bar{\Delta} = \Delta_0 e^{-2i\theta} \right] \right), \tag{4}$$

where

$$\hat{\mathcal{G}}^{-1} = \begin{pmatrix} \partial_\tau + i\phi + \frac{1}{2m}(-i\nabla - \boldsymbol{A})^2 - \mu & \Delta_0 e^{2i\theta} \\ \Delta_0 e^{-2i\theta} & \partial_\tau - i\phi - \frac{1}{2m}(-i\nabla + \boldsymbol{A})^2 + \mu \end{pmatrix} \tag{5}$$

is the Gor'kov's Greens's function. Since the theory is invariant under gauge transformations, first gauge away the phase dependence of the order parameter field, by using the unitary transformation $\hat{U} \equiv \text{diag}\left(e^{-i\theta}, e^{i\theta}\right)$ and then, using $\theta = 0, \phi^{'} = \phi + \partial_\tau \theta, \boldsymbol{A} + \boldsymbol{\nabla}\theta$ show that

$$\mathcal{G}^{-1} = U\mathcal{G}^{-1}U^\dagger = \underbrace{\tau_0\partial_\tau + \tau_3\left(-\frac{1}{2m}\nabla^2 - \mu\right) + \tau_1\Delta_0}_{\hat{\mathcal{G}}_0^{-1}} + \underbrace{i\tau_3\phi' + \frac{i}{2m}\tau_0\{\nabla, \boldsymbol{A}^{'}\}}_{\hat{\mathcal{X}}_1} + \underbrace{\tau_3\frac{1}{2m}\boldsymbol{A}'^2}_{\hat{\mathcal{X}}_2}, \quad (6)$$

where τ_j are the Pauli matrices in Nambu space.

c) Now expand the Tr ln and express the effective action as **2**pt(s)

$$S_{EM} \sim \text{const.} - \text{Tr}\left(\mathcal{G}_0\mathcal{X}_1\right) - \text{Tr}\left(\mathcal{G}_0\mathcal{X}_2\right) + \frac{1}{2}\text{Tr}\left(\mathcal{G}_0\mathcal{X}_1\mathcal{G}_0\mathcal{X}_1\right). \quad (7)$$

Now given that the first order term $\text{Tr}(\mathcal{G}_0\mathcal{X}_1) \propto in \int d\tau d^d r \phi(\tau, \mathbf{r})$, is zero, show that the second order term $\propto \mathcal{X}_2$ is given as

$$-\text{Tr}\left(\mathcal{G}_0\mathcal{X}_2\right) = \frac{n}{2m}\int d\tau \int d^d\boldsymbol{r}\boldsymbol{A}^2(\tau, \boldsymbol{r}), \quad (8)$$

where n is the density of electrons.

d) Finally, show that **3**pt(s)

$$\frac{1}{2}\text{Tr}\left(\mathcal{G}_0\mathcal{X}_1\mathcal{G}_0\mathcal{X}_1\right) = C_1\sum_q \phi'_q\phi'_{-q} + C_2\sum_q \boldsymbol{A}'_q \cdot \boldsymbol{A}'_q, \quad (9)$$

where

$$\mathcal{C}_1 = \sum_p \frac{\omega_n^2 - E_{\boldsymbol{p}}^2 + 2\Delta_0^2}{\left(\omega_n^2 + E_{\boldsymbol{p}}^2\right)^2}, \quad \mathcal{C}_2 = \sum_p \frac{1}{dm^2}\frac{\boldsymbol{p}^2\left(-\omega_n^2 + E_{\boldsymbol{p}}^2\right)}{\left(\omega_n^2 + E_{\boldsymbol{p}}^2\right)^2}. \quad (10)$$

Hint: You might want to use the relation $\sum_{\boldsymbol{p}}(\boldsymbol{p}\cdot\boldsymbol{v})(\boldsymbol{p}\cdot\boldsymbol{v}')F(\boldsymbol{p}^2) = \frac{\boldsymbol{v}\cdot\boldsymbol{v}'}{d}\sum_{\boldsymbol{p}}\boldsymbol{p}^2F(\boldsymbol{p}^2)$ that holds for rotationally invariant functions $F(\boldsymbol{p}^2)$. Note that we are working in d dimensions.

Problem 8.2: Coefficients of the effective Lagrangian

[**Written** | 5 pt(s)]

ID: ex_em_super2:qft25

Learning objective

We will analyze the obtained effective action, namely the coefficients entering the Lagrangian and their temperature dependence.

In the previous task, we have obtained the effective action describing coupling between the phase of the superconducting order parameter and the electromagnetic field,

$$S_{\text{EM}} = \int d\tau \int d^d\boldsymbol{r}\left[\mathcal{C}_1\left(\partial_\tau\theta + \phi\right)^2 + \left(\frac{n}{2m} + \mathcal{C}_2\right)\left(\nabla\theta - \boldsymbol{A}\right)^2\right], \quad (11)$$

where

$$\mathcal{C}_1 = \frac{T}{L^d}\sum_p \frac{\omega_n^2 - E_{\boldsymbol{p}}^2 + 2\Delta_0^2}{\left(\omega_n^2 + E_{\boldsymbol{p}}^2\right)^2}, \quad \mathcal{C}_2 = \frac{T}{L^d}\sum_p \frac{1}{dm^2}\frac{\boldsymbol{p}^2\left(-\omega_n^2 + E_{\boldsymbol{p}}^2\right)}{\left(\omega_n^2 + E_{\boldsymbol{p}}^2\right)^2}. \quad (12)$$

Here we will focus on calculating the coefficients $\mathcal{C}_1, \mathcal{C}_2$.

a) In the expressions for $\mathcal{C}_1, \mathcal{C}_2$ perform the summation over the fermionic Matsubara frequencies. **2**pt(s)

b) To perform the momentum integration, consider the two limiting cases: $\Delta_0/T \gg 1$ and $\Delta_0/T \ll 1$ (with $\Delta_0 \neq 0$). Try to come up with the physical interpretation of the obtained coefficients and their temperature dependence. **3**pt(s)

Hint: You may assume that there is always a well-defined Fermi surface around which some of the integrands will be peaked. In this case the sum over momenta can be approximated by $\frac{1}{L^d}\sum_{\boldsymbol{p}} f(\xi_{\boldsymbol{p}}) \approx \frac{\rho_F}{2}\int d\xi f(\xi)$, where ρ_F denotes the density of states at the Fermi energy and the extra factor of $1/2$ originates in the spin degeneracy.