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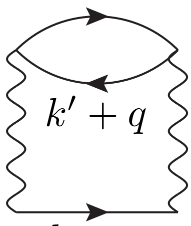
Problem 6.1: Self-energy diagram in fermi liquid theory [**Oral** | 6 pt(s)]

ID: ex_self_energy_fermi_liquid:qft25

Learning objective

In this exercise you will investigate one of the lowest-order diagrams that contributes to the imaginary part of the fermionic self-energy. This is particularly important in Fermi Liquid theory, for example, where the inverse of the imaginary part of the self-energy is related to the lifetime of quasiparticles.

The diagram



plays a central role in the microscopic theory of Fermi liquids. Here, the straight line represents the fermionic single-particle Green's function which we take to be of the form

$$G(i\omega_n, \boldsymbol{k}) = \frac{Z_{\boldsymbol{k}}}{i\omega_n - \epsilon_{\boldsymbol{k}}}, \qquad 0 < Z_{\boldsymbol{k}} \leq 1. \tag{1}$$

Naturally, the retarded and advanced expressions are then given by

$$G^{\mathrm{R}}(\omega, \boldsymbol{k}) = \frac{Z_{\boldsymbol{k}}}{\omega + i0^+ - \epsilon_{\boldsymbol{k}}} \quad \text{and} \quad G^{\mathrm{A}}(\omega, \boldsymbol{k}) = \frac{Z_{\boldsymbol{k}}}{\omega - i0^+ - \epsilon_{\boldsymbol{k}}}. \tag{2}$$

For simplicity, you can assume the dispersion to be parabolic, $\epsilon_{\boldsymbol{k}} = \boldsymbol{k}^2/(2m)$, although this is not necessary to solve the following problems. Furthermore, the four-vector notation $k = (i\omega_n, \boldsymbol{k})$ (fermionic) and $q = (i\Omega_n, \boldsymbol{q})$ (bosonic) combines Matsubara frequencies and momenta. The wiggly lines in the diagram refer to the four-fermion-interaction amplitude $U_{\boldsymbol{q}}$ that in general depends on the transferred momentum $\boldsymbol{q}$.

a) In order to understand the physical meaning of $Z_{\boldsymbol{k}}$, often referred to as "quasiparticle residue" [1], calculate the spectral function and the occupation number from **1** pt(s)

$$A(\omega, \boldsymbol{k}) = -\frac{1}{\pi}\mathrm{Im}G^{\mathrm{R}}(\omega, \boldsymbol{k}) \quad \text{and} \quad n_{\boldsymbol{k}} = \int_{-\infty}^{\infty} d\omega A(\omega, \boldsymbol{k}) n_F(\omega). \tag{3}$$

with $n_F(\omega)$ representing the Fermi-Dirac distribution. How do you interpret these quantities?

[1] Note: This quantity can be obtained experimentally with angle-resolved photoemission spectroscopy (ARPES). See, e.g., Chang, Johan, et al. "Anisotropic breakdown of Fermi liquid quasiparticle excitations in overdoped $La_{2-x}Sr_xCuO_4$." Nature communications 4.1 (2013): 2559.

b) Following the Feynman rules discussed in the lecture, argue why the analytical form (in Matsubara formalism) for the self-energy is given by 1pt(s)

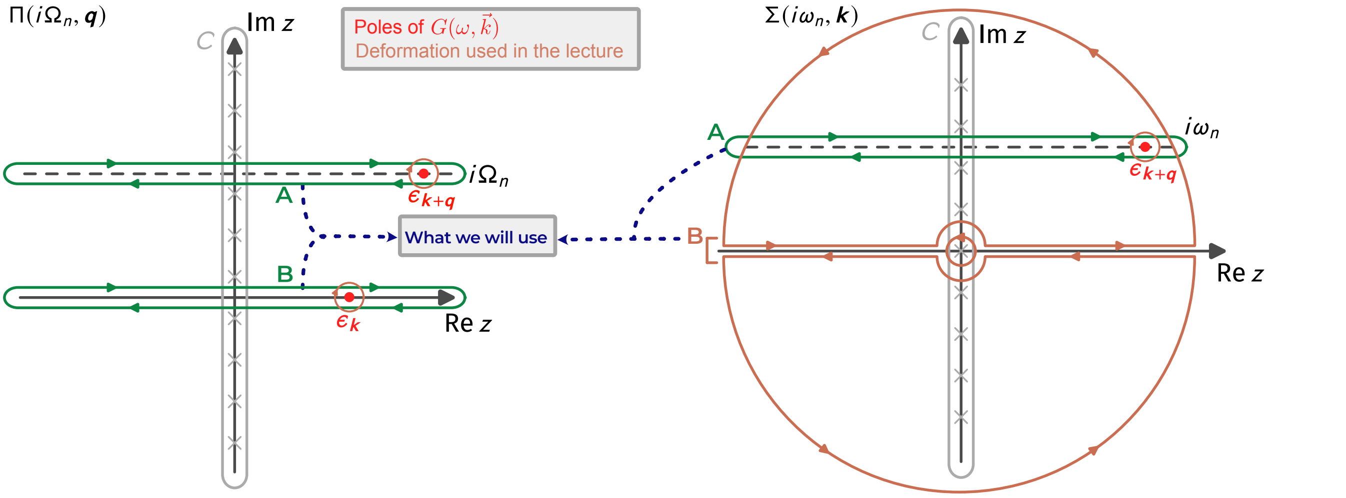
$$\Sigma\left(i\omega_n,\boldsymbol{k}\right)=T\sum_{\Omega_m}\int\frac{d^d\boldsymbol{q}}{(2\pi)^d}U_{\boldsymbol{q}}^2G\left(i\omega_n+i\Omega_m,\boldsymbol{k}+\boldsymbol{q}\right)\Pi\left(i\Omega_m,\boldsymbol{q}\right), \tag{4}$$

where d is the dimensionality of the system. Likewise, write the analytical form (also in the Matsubara formalism) of the particle-hole bubble $\Pi(i\Omega_n,\boldsymbol{q})$ in terms of Eq. (1).[2]

For the following two tasks, remember that

$$2T=\begin{cases}\mathrm{Res}\left[\coth(\beta z/2),z=i\Omega_n\right], & \Omega_n=2\pi nT\ \text{(Bosons)}\\ \mathrm{Res}\left[\tanh(\beta z/2),z=i\omega_n\right], & \omega_n=2\pi(n+1)T\ \text{(Fermions)}\end{cases}, \tag{5}$$

for $n\in\mathbb{Z}$, and consider the contours from the following figures:



c) Using the residue theorem (according to (5)), the contour in the complex plane shown above, and subsequent analytic continuation $i\omega_n\to\omega+i0^+$ to the real axis, show that the (retarded) particle-hole bubble expression can be rewritten as 2pt(s)

$$\begin{aligned}\Pi^R(\Omega,\boldsymbol{q})=2\int\frac{\mathrm{d}^d\boldsymbol{k}}{(2\pi)^d}\int\frac{\mathrm{d}\omega}{2\pi}\Bigg[&\tanh\left(\frac{\omega}{2T}\right)G^R(\omega+\Omega,\boldsymbol{k}+\boldsymbol{q})\,\mathrm{Im}G^R(\omega,\boldsymbol{k})\\&+\tanh\left(\frac{\omega+\Omega}{2T}\right)G^A(\omega,\boldsymbol{k})\,\mathrm{Im}G^R(\omega+\Omega,\boldsymbol{k}+\boldsymbol{q})\Bigg]\end{aligned} \tag{6}$$

Hint: The results we obtained in Problem 3.2a) will be helpful here :)

d) Similarly, considering the second contour from the image above and the definition Eq. (4), show that the (retarded) self-energy expression can be rewritten as 2pt(s)

$$\begin{aligned}\Sigma^R(\omega,\boldsymbol{k})=\int\frac{\mathrm{d}^d\boldsymbol{q}}{(2\pi)^d}U_{\boldsymbol{q}}^2\Bigg[&\mathcal{P}\int\frac{\mathrm{d}\Omega}{2\pi}\coth\left(\frac{\Omega}{2T}\right)G^R(\omega+\Omega,\boldsymbol{k}+\boldsymbol{q})\,\mathrm{Im}\Pi^R(\Omega,\boldsymbol{q})\\&+\int\frac{\mathrm{d}\Omega}{2\pi}\tanh\left(\frac{\Omega+\omega}{2T}\right)\Pi^A(\Omega,\boldsymbol{q})\,\mathrm{Im}G^R(\omega+\Omega,\boldsymbol{k}+\boldsymbol{q})\Bigg],\end{aligned} \tag{7}$$

[2] There is a factor of 2 here coming from the summation over the spin.

with $\mathcal{P}\int$ denoting the principle value integral, since there is a pole at $z = 0$.

Problem 6.2: Self-energy diagram in fermi liquid theory II [**Written** | 4 pt(s)]

ID: ex_self_energy_fermi_liquid_written:qft25

Learning objective

This is the written continuation from the previous exercise.

a) Let us now focus on the imaginary part of $\Sigma^R(\omega, \boldsymbol{k})$. First, the particle-hole bubble $\Pi^R(\Omega, \boldsymbol{q})$ contribution comes only from its imaginary part $\mathrm{Im}\Pi^R(\Omega, \boldsymbol{q})$ (why?). **2**pt(s)

Focusing on small T and ω (compared to the Fermi energy E_F), which allows neglecting ω and Ω in the delta functions appearing in the expression for $\mathrm{Im}\Pi^R(\Omega, \boldsymbol{q})$, show that

$$\mathrm{Im}\Pi^R(\Omega, \boldsymbol{q}) \sim A_{\boldsymbol{q}}\,\Omega \tag{8}$$

and that the explicit form of the prefactor $A_{\boldsymbol{q}}$ is given in terms of the quasiparticle residues as

$$A_{\boldsymbol{q}} = -\frac{1}{(2\pi)^{d-1}} \oint dS_{\boldsymbol{k}_F} Z_{\boldsymbol{k}_F} Z_{\boldsymbol{k}_F+\boldsymbol{q}} \delta\left(\varepsilon_{\boldsymbol{k}_F+\boldsymbol{q}}\right)\Big|_{\varepsilon_{\boldsymbol{k}}=0}, \tag{9}$$

where $dS_{\boldsymbol{k}}$ is a Fermi surface element.

Hint: To obtain Eq. (8) you can take the density of states to be independent of the direction normal to the Fermi surface. Additionally, note that

$$\coth\left(\frac{x}{2}\right) = 2n_B(x) + 1 \text{ and } \tanh\left(\frac{x}{2}\right) = 1 - 2n_F(x), \tag{10}$$

where the Fermi-Dirac ($\zeta = +1$) and Bose-Einstein ($\zeta = -1$) distributions are given by $n_\zeta(x) = \frac{1}{e^x + \zeta}$.

b) Using this result, obtain **2**pt(s)

$$\mathrm{Im}\Sigma^R(\omega, \boldsymbol{k}) \sim B_{\boldsymbol{k}}\left(\omega^2 + \pi^2 T^2\right) \tag{11}$$

for $\omega, T \ll E_F$. This is the typical behavior of a Fermi liquid. Show that the prefactor $B_{\boldsymbol{k}}$ is defined in terms of the quasiparticle residues as

$$B_{\boldsymbol{k}} = -\frac{1}{2}\frac{1}{(2\pi)^{d-1}} \int \frac{d^d\boldsymbol{q}}{(2\pi)^d} U_{\boldsymbol{q}}^2 \oint dS_{\boldsymbol{k}'_F} Z_{\boldsymbol{k}'_F} Z_{\boldsymbol{k}'_F+\boldsymbol{q}} Z_{\boldsymbol{k}_F+\boldsymbol{q}} \delta\left(\varepsilon_{\boldsymbol{k}'_F+\boldsymbol{q}}\right) \delta\left(\varepsilon_{\boldsymbol{k}_F+\boldsymbol{q}}\right). \tag{12}$$