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Problem 5.1: Polarization operator for graphene

[**Oral** | 4 pt(s)]

ID: ex_polarization_bubble_graphene:qft25

Learning objective

The Random Phase Approximation (RPA), in which the bare interaction is renormalized into an effective interaction, plays a central role in many-body systems, giving rise to key phenomena such as screening. In this exercise, you will explore the polarization operator—the principal quantity needed to evaluate screening effects within the RPA framework.

In this exercise we will calculate the retarded polarization operator $\Pi^R(\Omega, \boldsymbol{q})$ of a single Dirac cone of graphene, i.e., of a 2D system with Bloch Hamiltonian

$$h_{\boldsymbol{k}} = v_F(\sigma_x k_x + \sigma_y k_y). \tag{1}$$

Here σ_x and σ_y are Pauli matrices in pseudospin space.

a) Show that the Matsubara Green's function of the system can be written as **2**pt(s)

$$G_p = \sum_{\lambda=\pm} \frac{P_{\lambda,\boldsymbol{p}}}{-i\omega_n + \lambda v_F |\boldsymbol{p}|}, P_{\lambda,\boldsymbol{p}} = \frac{1}{2}\left(\sigma_0 + \lambda \boldsymbol{\sigma} \frac{\boldsymbol{p}}{|\boldsymbol{p}|}\right). \tag{2}$$

What is the meaning of the operators $\mathcal{P}_{\lambda,\boldsymbol{p}}$?

b) On the imaginary axis, the polarization operator is given by **2**pt(s)

$$\Pi(i\Omega_n, \boldsymbol{q}) = T \sum_{\omega_n} \int \frac{\mathrm{d}^2\boldsymbol{k}}{(2\pi)^2} \mathrm{Tr}\left[G(i\omega_n, \boldsymbol{k}) G(i\omega_n + i\Omega_n, \boldsymbol{k}+\boldsymbol{q})\right] \tag{3}$$

with Ω_n and ω_n denoting bosonic and fermionic Matsubara frequencies, respectively. Evaluate the trace $\mathrm{Tr}[\dots]$, over pseudospin space using the Green's function representation given in equation (2) and perform the Matsubara summation to demonstrate that

$$\Pi\left(i\Omega_n, \boldsymbol{q}\right) = \frac{1}{\Omega} \sum_{\lambda,\lambda'} \sum_{\boldsymbol{p}} \frac{1}{2}\left[1 + \lambda\lambda' \cos(\sphericalangle(\boldsymbol{p}, \boldsymbol{p}+\boldsymbol{q}))\right] \times \frac{n_F\left(\epsilon_\lambda(\boldsymbol{p})\right) - n_F\left(\epsilon_{\lambda'}\left(\boldsymbol{p}+\boldsymbol{q}\right)\right)}{i\Omega_m + \epsilon_\lambda(\boldsymbol{p}) - \epsilon_{\lambda'}(\boldsymbol{p}+\boldsymbol{q})}, \tag{4}$$

where $\sphericalangle(\boldsymbol{a}, \boldsymbol{b})$ is the angle between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

Problem 5.2: Polarization operator of graphene (continued)

[**Written** | 7 (+4 bonus) pt(s)]

ID: ex_polarization_bubble_graphene_2:qft25

Learning objective

This is a continuation of the previous exercise. We will now focus on the zero temperature limit and calculate the integration over the momentum for the polarization operator explicitly.

a) In the limit $T \to 0$, identify the relevant terms in the expression for the polarization operator from the previous exercise and perform an analytic continuation $\Pi(i\Omega_n, \boldsymbol{q}) \to \Pi^R(\Omega, \boldsymbol{q})$ to show that **2pt(s)**

$$\Pi^R(\Omega, \boldsymbol{q}) = + \int \frac{d^2\boldsymbol{p}}{(2\pi)^2}(1 - \cos(\sphericalangle(\boldsymbol{p}, \boldsymbol{p}+\boldsymbol{q}))) \times \frac{v_F(|\boldsymbol{p}| + |\boldsymbol{p}+\boldsymbol{q}|)}{(\Omega + i0^+)^2 - [v_F(|\boldsymbol{p}| + |\boldsymbol{p}+\boldsymbol{q}|)]^2}. \quad (5)$$

Hint: Remember that at $T = 0$

$$n_F(\epsilon_\lambda(\boldsymbol{p})) = \begin{cases} 0 & \lambda = + \\ 1 & \lambda = - \end{cases} \quad (6)$$

and that an analytic continuation can be performed via $i\Omega_n \to \Omega + i0^+$.

b) To perform the momentum integration, it is convenient to introduce new variables ξ and η via **3pt(s)**

$$\xi = |\boldsymbol{p}| + |\boldsymbol{p}+\boldsymbol{q}|, \qquad \eta = |\boldsymbol{p}| - |\boldsymbol{p}+\boldsymbol{q}|. \quad (7)$$

What is the geometric meaning of ξ and η? In order to rewrite the momentum integration, show that

$$\cos(\theta_{\boldsymbol{p},\boldsymbol{p}+\boldsymbol{q}}) = \frac{2\boldsymbol{q}^2 - \eta^2 - \xi^2}{\eta^2 - \xi^2} \quad (8)$$

with $\theta_{\boldsymbol{p},\boldsymbol{p}+\boldsymbol{q}} = \sphericalangle(\boldsymbol{p}, \boldsymbol{p}+\boldsymbol{q})$ denoting the angle between $\boldsymbol{p}$ and $\boldsymbol{p}+\boldsymbol{q}$ and that

$$\mathrm{d}p_x \mathrm{d}p_y = \frac{\xi^2 - \eta^2}{4\sqrt{(\boldsymbol{q}^2 - \eta^2)(\xi^2 - \boldsymbol{q}^2)}} \mathrm{d}\xi \mathrm{d}\eta. \quad (9)$$

You might find it useful to choose $\boldsymbol{q} = (q_x, 0)$ which is possible without loss of generality.

c) Rewrite the momentum integral in terms of ξ and η and perform the integrations using the identity **2pt(s)**

$$\int_1^\infty \mathrm{d}x \frac{x}{\sqrt{x^2 - 1}} \frac{1}{w^2 - x^2} = \frac{-\pi}{2\sqrt{1 - w^2}} \quad (10)$$

for $w \in \mathbb{C}$ (except for $w \in \mathbb{R}$ with $|w| \geq 1$).

*d) Sketch $\mathrm{Re}\Pi^R(\Omega, \boldsymbol{q})$ and $\mathrm{Im}\Pi^R(\Omega, \boldsymbol{q})$ as a function of Ω. **+2pt(s)**

*e) Check your result for $\mathrm{Im}\Pi^R(\Omega, \boldsymbol{q})$ by using $\frac{1}{(x+i0^+)} = \mathcal{P}\frac{1}{x} - i\pi\delta(x)$ before performing the ξ integration. **+2pt(s)**