Prof. Dr. Mathias Scheurer
Institute for Theoretical Physics III, University of Stuttgart

April 25th, 2025
SS 2025

Problem 4.1: Warm-up for the ϕ^4-theory

[**Written** | 6 pt(s)]

ID: ex_lambdaphi4_greenf:qft25

> **Learning objective**
>
> The purpose of this problem is to become familiar with the non-interacting quadratic theory before applying the machinery of Feynman diagrams later on to the ϕ^4 theory.

We first consider a quadratic field theory defined by

$$\mathcal{Z} \equiv \int D\phi e^{-S[\phi]}, \quad S_0[\phi] \equiv \int d^d x \left(\frac{1}{2}(\partial\phi)^2 + \frac{r}{2}\phi^2 \right) \tag{1}$$

where ϕ is a real scalar bosonic field. The propagator of the theory is given by

$$G_0(\boldsymbol{x}_1, \boldsymbol{x}_2) = \langle \phi(\boldsymbol{x}_1)\phi(\boldsymbol{x}_2) \rangle_0 \,. \tag{2}$$

a) Express the Gaussian action S_0 in a momentum representation, using the following: **2**pt(s)

$$\phi(\boldsymbol{x}) = \frac{1}{\sqrt{\Omega}} \sum_{\boldsymbol{k}} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \phi_{\boldsymbol{k}}, \tag{3}$$

where Ω represents the volume of the system.

b) Using the expression **2**pt(s)

$$\langle \cdots \rangle \equiv (2\pi)^{-N/2} \det \boldsymbol{A}^{1/2} \int d\boldsymbol{v} e^{-\frac{1}{2}\boldsymbol{v}^T \boldsymbol{A} \boldsymbol{v}} (\cdots), \tag{4}$$

show that the propagator can be expressed as,

$$G_0(\boldsymbol{x}_1, \boldsymbol{x}_2) \approx \int \frac{d^d k}{(2\pi)^d} \frac{e^{-i\boldsymbol{k}\cdot(\boldsymbol{x}_1 - \boldsymbol{x}_2)}}{\boldsymbol{k}^2 + r}. \tag{5}$$

Hint: Assume that the system size is large in order to take the continuum limit.

c) Finally, show that for a one-dimensional system, the propagator is given by, **2**pt(s)

$$G_0(x) = \frac{1}{2\sqrt{r}} e^{-\sqrt{r}|x|}. \tag{6}$$

where you can assume without loss of generality: $x_1 = x, x_2 = 0$.

Problem 4.2: Pertubation theory for the ϕ^4-theory

[**Oral** | 10 pt(s)]

ID: ex_lambdaphi4_perturbation:qft25

Learning objective

The ϕ^4-theory is a paradigmatic example where we can make use of perturbation theory with Feynman diagrams to get insights into physical observables. More specifically, in this exercise we will see how to expand the partition function in terms of contractions over fields in order to get an estimate of corrections to the free energy of the system.

We consider again a quadratic field theory defined by

$$\mathcal{Z} \equiv \int D\phi e^{-S[\phi]}, \quad S[\phi] \equiv \int d^d x \left(\frac{1}{2}(\partial\phi)^2 + \frac{r}{2}\phi^2 + g\phi^4\right) \tag{7}$$

where ϕ is a scalar bosonic field.

a) Recall the path integral formulation of the many body partition function and expand it up to second order in g. Express the expansion in terms of Feynman diagrams and find the corresponding symmetry factors. **2pt(s)**

Hint: You will need to use Wick's theorem to decouple terms like

$$\langle\phi(1)\phi(2)\phi(3)\phi(4)\rangle_0 = \langle\phi(1)\phi(2)\rangle_0\langle\phi(3)\phi(4)\rangle_0 + \langle\phi(1)\phi(3)\rangle_0\langle\phi(2)\phi(4)\rangle_0 + \tag{8}$$

$$+ \langle\phi(1)\phi(4)\rangle_0\langle\phi(2)\phi(3)\rangle_0 \tag{9}$$

b) Calculate the free energy of the system $F = -T\ln Z$ with the result obtained above up to second order in g. Show that the disconnected diagrams in the expansion are cancelled out. **2pt(s)**

Note: This holds up to arbitrary order in the g expansion and is called *Linked Cluster Theorem.*

c) Show that for a 10th-order contraction there will be $945 = (10-1)!!$ terms given by **4pt(s)**

$$\left\langle\phi(\mathbf{x})\phi(\mathbf{y})^4\phi\left(\mathbf{y}'\right)^4\phi\left(\mathbf{x}'\right)\right\rangle_0 = 9G_0\left(\mathbf{x}-\mathbf{x}'\right)G_0(\mathbf{0})^4 + 72G_0\left(\mathbf{x}-\mathbf{x}'\right)G_0\left(\mathbf{y}-\mathbf{y}'\right)^2 G_0(\mathbf{0})^2 \tag{10}$$

$$+ 24G_0\left(\mathbf{x}-\mathbf{x}'\right)G_0\left(\mathbf{y}-\mathbf{y}'\right)^4 + \left[36G_0(\mathbf{x}-\mathbf{y})G_0\left(\mathbf{x}'-\mathbf{y}\right)G_0(\mathbf{0})^3\right. \tag{11}$$

$$+ 144\left(G_0(\mathbf{x}-\mathbf{y})G_0\left(\mathbf{x}'-\mathbf{y}\right)G_0\left(\mathbf{y}-\mathbf{y}'\right)^2 G_0(\mathbf{0}) +\right. \tag{12}$$

$$\left. + G_0(\mathbf{x}-\mathbf{y})G_0\left(\mathbf{x}'-\mathbf{y}'\right)G_0(\mathbf{0})^2 G_0\left(\mathbf{y}-\mathbf{y}'\right)\right) \tag{13}$$

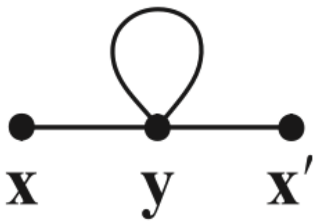
$$\left. + 96G_0(\mathbf{x}-\mathbf{y})G_0\left(\mathbf{x}'-\mathbf{y}'\right)G_0\left(\mathbf{y}'-\mathbf{y}\right)^3 + \left(\mathbf{y}\leftrightarrow\mathbf{y}'\right)\right] \tag{14}$$

where

$$G_0(\boldsymbol{x}_1, \boldsymbol{x}_2) = G_0(\boldsymbol{x}_1 - \boldsymbol{x}_2) = \langle\phi(\boldsymbol{x}_1)\phi(\boldsymbol{x}_2)\rangle_0 . \tag{15}$$

Identify each term with their corresponding diagram and explain how you obtained the symmetry factors. Which diagrams will be left after the use of the Linked Cluster Theorem?

d) The previous calculations could also have been done in momentum-space, and the connection between both versions is quite natural. Show that the momentum-space representation of the following diagram **2pt(s)**



,

for example, is related to the one in real-space by a Fourier transformation.

Hint: Use the expressions you calculated for the non-interacting Green's function in the first exercise and the fact that all momenta flowing into a vertex are equal to zero. The last statement is know as *Kirchhoff's law*.