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Problem 2.1: Path integral of the partition function [**Oral** | 2 pt(s)]

ID: ex_path_integral_partition_function:qft25

> **Learning objective**
>
> Previously, we have seen how the transition amplitude $\mathcal{T}_{(\boldsymbol{x_i},t_i)\to(\boldsymbol{x_f},t_f)}$ can be expressed as a path integral. In this formulation, the action in the exponent appears with an extra factor of i which can lead to some convergence problems. In this exercise, you'll derive the path integral formulation of the partition function for a single particle through a Wick rotation, which connects quantum mechanics to statistical physics and resolves the convergence problem.

Show that the partition function $Z = \mathrm{Tr}\left[e^{-\beta\hat{H}}\right]$ can be expressed as

$$Z \propto \int_{\boldsymbol{x}(\beta)=\boldsymbol{x}(0)} D\boldsymbol{x} e^{-\int_0^\beta d\tau\left(\frac{1}{2}m(\partial_\tau \boldsymbol{x})^2 + U(\boldsymbol{x})\right)}. \tag{1}$$

The Hamiltonian $\hat{H} = \hat{T}(\boldsymbol{p}) + \hat{U}(\boldsymbol{x})$ can be separated in a kinetic term $\hat{T}$ and a potential term $\hat{U}$.

Problem 2.2: The harmonic oscillator propagator in the path integral formalism [**Oral** | 6 (+2 bonus) pt(s)]

ID: ex_harmonic_oscillator_path_integral_formulation:qft25

> **Learning objective**
>
> After becoming acquainted with the Path integral formalism, we'll apply this technique to the harmonic oscillator in our final example, deriving both its propagator and partition function.

The propagator for a particle of mass m in a *one-dimensional* harmonic potential $V(x) = \frac{1}{2}m\omega^2x^2$ is given by

$$\begin{aligned}\mathcal{T}_{(x_i,t_i)\to(x_f,t_f)} = iG\left(x_f,t_f;x_i,t_i\right) = \sqrt{\frac{m\omega}{2\pi i\sin(\omega T)}} \times \\ \exp\left\{\frac{im\omega}{2\sin(\omega T)}\left[(x_i^2+x_f^2)\cos(\omega T) - 2x_ix_f\right]\right\}\,,\end{aligned} \tag{2}$$

where $T = t_f - t_i$. Without loss of generality, you can consider $T = t_f$.

a) In the path integral formulation, the propagator is calculated from the action $S[x]$ as **2**pt(s)

$$G\left(x_f,t_f;x_i,t_i\right) = \int_{x_i}^{x_f} \mathcal{D}x\; e^{iS[x]} \tag{3}$$

where $x : [t_i, t_f] \to \mathbb{R}$ denotes trajectories of the particle with $x(t_i) = x_i$ and $x(t_f) = x_f$.

Express these trajectories as $x(t) = \overline{x}(t) + y(t)$, i.e., a sum of the classical path $\overline{x}(t)$ and fluctuations $y(t)$. Write the action as the sum of the classical action and the contribution of the fluctuations.

What are the boundary conditions for the fluctuations?

Show that

$$G\left(x_f, t_f; x_i, t_i\right) = F(T)\, e^{iS[\overline{x}]} , \tag{4}$$

and demonstrate that $F(T)$ is independent of the initial and final positions x_i and x_f.

b) Show that the classical solution of the harmonic oscillator takes the form **1**pt(s)

$$\overline{x}(t) = \frac{x_f - x_i \cos(\omega t_f)}{\sin(\omega t_f)} \sin(\omega t_i) + x_i \cos(\omega t_i) . \tag{5}$$

Evaluate the classical action of this solution to obtain the factor $e^{iS[\overline{x}]}$.

*c) Show that the prefactor is given by **+2**pt(s)

$$F(t_f) = \sqrt{\frac{m\omega}{2\pi i}} \frac{1}{\sqrt{\sin(\omega t_f)}} \tag{6}$$

using the expansion of the fluctuations

$$y(t) = \sum_{n=1}^{\infty} a_n y_n(t) , \tag{7}$$

with $y_n(t) = \sqrt{\frac{2}{t_f}} \sin(n\pi t/t_f)$.

Hint:

- Use the eigenvalue equation $(-\partial_t^2 - \omega^2) y_n(t) = \lambda_n y_n(t)$ and the orthonormality of y_n to evaluate $S[y]$.
- Use the parametrization $\mathcal{D}y = J \prod_{n=1}^{\infty} \mathrm{d}a_n$, where J is a (yet) undetermined normalization constant. Calculate $F(t_f)$ as a function of J and λ_n.
- To get rid of J, study the limit $\omega \to 0$ of the propagator. In this limit, one must obtain the solution for a free particle and can derive $F_0(t_f) = \lim_{\omega \to 0} F(t_f)$. Use this finding to obtain the result for arbitrary ω as $F(t_f) = \frac{F(t_f)}{F_0(t_f)} F_0(t_f)$. The fraction $\frac{F(t_f)}{F_0(t_f)}$ can be simplified using the identity $\sin(x) = x \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2\pi^2}\right)$.

Now that we have the propagator for the harmonic oscillator, we can take a look at how we can use this quantity to calculate the partition function from statistical mechanics.

d) As a first reminder, obtain the partition function for this system using **1**pt(s)

$$Z = \mathrm{tr}\, e^{-\beta \hat{H}} \tag{8}$$

in the eigenbasis of $\hat{H}$.

e) Consider paths with $x = x_f = x_i$ and obtain the partition function for the system, now from the path integral perspective. **2**pt(s)

Write Z in a similar form to (8). What is the conceptual modification on time t necessary to connect your result to Z in this formalism?

Problem 2.3: Gaussian integration with Grassmann variables

[**Written** | 2 pt(s)]

ID: ex_path_integral_grassmann:qft25

> **Learning objective**
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> Having mastered Gaussian integrals over real and complex variables, we now turn to *Grassmann variables.* Understanding these anticommuting variables and their integration properties is crucial for formulating fermionic path integrals.

Let us first recall the defining properties of Grassmann numbers,

$$\psi_\alpha \psi_\beta = -\psi_\beta \psi_\alpha, \tag{1}$$

$$\int d\psi = 0, \tag{2a}$$

$$\int d\psi \, \psi = 1. \tag{2b}$$

Using these properties, show that

$$\int \prod_{\alpha=1}^{N} d\psi'_\alpha d\psi_\alpha e^{-\psi'_\alpha A_{\alpha\beta}\psi_\beta + f'_\alpha \psi_\alpha + \psi'_\alpha f_\alpha} = \det(A) e^{f'_\alpha A^{-1}_{\alpha\beta} f_\beta}, \tag{9}$$

where $\psi_\alpha, \psi'_\alpha \in \mathcal{A}$ are Grassmann variables from the Grassmann algebra $\mathcal{A}$, and A is a complex hermitian matrix.

Hint: Follow the same strategies in the previous list, i.e., solve the unshifted Gaussian integral to find that it is given by $\det(A)$, and argue that shifting the integration variables as

$$\psi_\alpha \to \psi_\alpha - A^{-1}_{\alpha\beta} f_\beta$$
$$\psi'_\alpha \to \psi'_\alpha - f'_\beta A^{-1}_{\beta\alpha}$$

does not change the result, in order to get the additional term $e^{f'_\alpha A^{-1}_{\alpha\beta} f_\beta}$.