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Information on lecture and tutorials

Here a few infos on the modalities of the course ***"Quantum Field Theory"***:

- The C@MPUS-ID of this course is 045570002.
- You can find detailed information on lecture and tutorials on the website of our institute:

  https://itp3.info/qft25

- **Written** problems have to be handed in via ILIAS and will be corrected by the tutors. You must earn at least **50 %** of the written points to be admitted to the exam.
- **Oral** problems have to be prepared for the exercise session and will be presented by a student at the blackboard. You must earn at least **66 %** of the oral points to be admitted to the exam.
- Every student is required to **present** at least **3** of the oral problems at the blackboard to be admitted to the exam.
- Problems marked with an asterisk ($*$) are optional and can earn you bonus points.
- If you have questions regarding the problem sets, feel free to contact your tutor at any time.

Problem 1.1: Gaussian integrals

[**Written** | 7 pt(s)]

ID: ex_gaussian_integrals_qft:qft25

Learning objective

In this exercise we will systematically explore Gaussian integrals, beginning with the simplest one-dimensional case and progressing to the N-dimensional scenarios with complex variables. This approach will build a solid foundation for understanding the mathematical structure underlying quantum mechanical path integrals.

a) Use your favorite integration technique to solve the integral **1** pt(s)

$$\int_{-\infty}^{\infty} dx e^{-\frac{a}{2}x^2}. \tag{1}$$

What is the restriction on $a \in \mathbb{R}$ such that the integral converges?

b) Use the result of task a) to solve **1** pt(s)

$$\int_{-\infty}^{\infty} dx e^{-\frac{a}{2}x^2+bx}. \tag{2}$$

We now want to generalize this to the N-dimensional integration variable $\phi \in \mathbb{R}^N$.

c) Show that **2**pt(s)

$$\int_{\mathbb{R}^N} d^N\phi e^{-\frac{1}{2}\phi^T A\phi} = \frac{(2\pi)^{N/2}}{\sqrt{\det A}} \tag{3}$$

with $A \in \mathrm{GL}(N)$ and $A^T = A$. How does the restriction of a in task a) convert to A such that convergence is guaranteed?

d) Show that adding a source term $\phi \cdot \boldsymbol{j}$ with $\boldsymbol{j} \in \mathbb{R}^N$ leads to **1**pt(s)

$$\int_{\mathbb{R}^N} d^N\phi e^{-\frac{1}{2}\phi^T A\phi + \phi\cdot\boldsymbol{j}} = \frac{(2\pi)^{N/2}}{\sqrt{\det A}} e^{\frac{1}{2}\boldsymbol{j}^T A^{-1}\boldsymbol{j}}. \tag{4}$$

Hint: Argue that shifting the integration variable in Eq. (3) according to $\phi = \Phi - A^{-1}\boldsymbol{j}$ does not change the result of the integral and use this equality to show Eq. (4).

Finally we generalize the Gaussian integral to the N-dimensional complex case by integrating over $\phi \in \mathbb{C}^N$. To do so we introduce the notation $\int d(z^*, z)$ which is to be understood as an integration over the whole complex plane i.e. $\int d(z^*, z) \equiv \int_{-\infty}^{\infty} d\,\mathrm{Re}\, z \int_{-\infty}^{\infty} d\,\mathrm{Im}\, z$. The $1D$ Gaussian integral can be expanded to the complex plane as

$$\begin{aligned}\int d(z^*, z) e^{-wz^*z} &= \int_{-\infty}^{\infty} d\,\mathrm{Re}\, z \int_{-\infty}^{\infty} d\,\mathrm{Im}\, z e^{-wz^*z} \\ &= \int_{-\infty}^{\infty} da \int_{-\infty}^{\infty} db e^{-w(a-ib)(a+ib)} \\ &= \int_{-\infty}^{\infty} da \int_{-\infty}^{\infty} db e^{-w(a^2+b^2)} = \frac{\pi}{w} \qquad \text{for } \mathrm{Re}\, w > 0\end{aligned}$$

e) Show that **2**pt(s)

$$\int d(\phi^\dagger, \phi) e^{-\phi^\dagger A\phi + \boldsymbol{j}^\dagger\boldsymbol{\phi} + \phi^\dagger\boldsymbol{j'}} = \frac{\pi^N}{\det A} e^{\boldsymbol{j}^\dagger A^{-1}\boldsymbol{j'}} \tag{5}$$

where $\boldsymbol{j}, \boldsymbol{j'} \in \mathbb{C}^N$ and $A = A^\dagger$. In N dimensions the notation is to be understood as

$$\int d(\phi^\dagger, \phi) \equiv \int_{-\infty}^{\infty} d\mathrm{Re}\phi_1 \int_{-\infty}^{\infty} d\mathrm{Im}\phi_1 \int_{-\infty}^{\infty} d\mathrm{Re}\phi_2 \int_{-\infty}^{\infty} d\mathrm{Im}\phi_2 \cdots \int_{-\infty}^{\infty} d\mathrm{Re}\phi_N \int_{-\infty}^{\infty} d\mathrm{Im}\phi_N \tag{6}$$

Hint: In the complex case $\phi^\dagger = \Phi^\dagger - \boldsymbol{j}^\dagger A^{-1}$ and $\phi = \Phi - A^{-1}\boldsymbol{j'}$ can be shifted independently (note the $\boldsymbol{j}$ and $\boldsymbol{j'}$).

Problem 1.2: Free particle propagator with path integral

[**Written** | 4 pt(s)]

ID: ex_path_integral_free_particle:qft25

Learning objective

As a first application of path integrals, we will examine the canonical example of a free particle. Our objective is to derive the propagator—the same one typically obtained as the Green's function of the Schrödinger equation—by applying the Gaussian integration techniques developed in the previous exercise. This approach will demonstrate how path integrals provide an elegant alternative formulation of quantum mechanics.

Consider a free particle in one dimension with the Hamiltonian given by

$$\hat{H} = \frac{\hat{p}^2}{2m}, \tag{7}$$

with the usual momentum operator $\hat{p}$ and mass m. The propagator (see lecture) is given by

$$\mathcal{T}_{(x_i,t_i)\to(x_f,t_f)} = \int \mathcal{D}x\mathcal{D}p \exp\left[i\int_{t_i}^{t_f} \mathrm{d}t(\dot{x}(t)p(t) - H(p,x,t))\right], \tag{8}$$

which can be discretized as

$$\mathcal{T}_{(x_i,t_i)\to(x_i,t_f)} = \lim_{N\to\infty}\int \prod_{k=1}^{N-1} \mathrm{d}x_k \prod_{l=0}^{N-1} \frac{\mathrm{d}p_l}{2\pi} \exp\left[i\Delta t \sum_{\ell=0}^{N-1} \left(\dot{x}_\ell p_\ell - H\left(p_\ell, x_\ell, t_\ell\right)\right)\right], \tag{9}$$

with $\Delta t = \left(t_f - t_i\right)/N$.

a) By performing the Gaussian integrals over the momentum variables, show that each integral yields a normalization factor given by $\sqrt{\frac{m}{2\pi i\Delta t}}$. **1**pt(s)

b) Show that the free particle propagator is given by **3**pt(s)

$$\mathcal{T}_{(x_i,t_i)\to(x_f,t_f)} = \sqrt{\frac{m}{2\pi i\left(t_f - t_i\right)}} \exp\left[\frac{im\left(x_f - x_i\right)^2}{2\left(t_f - t_i\right)}\right] \tag{10}$$

using the path integral approach.