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Problem 8.1: Electromagnetic response of superconductors

[**Written** | 8 pt(s)]

ID: ex_em_super1:qft24

> **Learning objective**
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> In this exercise you will microscopically derive the effective action of the phase mode of a superconductor in presence of electromagnetic fields.

We consider a coupling to the vector potential $\boldsymbol{A}$ and scalar potential ϕ as

$$\int d^d r \int d\tau \bar{\psi}_\sigma \left(\partial_\tau + ie\phi + \frac{1}{2m}(-i\boldsymbol{\nabla} - e\boldsymbol{A})^2 - \mu \right) \psi_\sigma. \tag{1}$$

This coupling is known as minimal-coupling of an electromagnetic field. The full action can then be written as

$$S = \int d\tau d^d r \left(\bar{\psi}_\sigma \left(\partial_\tau + i\phi + \frac{1}{2m}(-i\nabla - \mathbf{A})^2 - \mu \right) \psi_\sigma - V \bar{\psi}_\uparrow \bar{\psi}_\downarrow \psi_\downarrow \psi_\uparrow \right) \tag{2}$$

The fields under gauge transformations behave as $\psi \to e^{i\theta}\psi$, $\psi^\dagger \to e^{-i\theta}\psi^\dagger$, $\phi \to \phi - e^{-1}\partial_\tau \theta$ and $\boldsymbol{A} \to \boldsymbol{A} + e^{-1}\boldsymbol{\nabla}\theta$.

a) Perform a Hubbard–Stratonovich transformation in the Cooper channel and show that **1**pt(s)

$$S = \int d^d r \int d\tau \left[\bar{\psi}_\sigma \left(\partial_\tau + ie\phi + \frac{1}{2m}(-i\boldsymbol{\nabla} - e\boldsymbol{A})^2 - \mu \right) \psi_\sigma + \left(\frac{1}{V}|\Delta|^2 + \left(\Delta \bar{\psi}_\uparrow \bar{\psi}_\downarrow + \bar{\Delta} \psi_\downarrow \psi_\uparrow \right) \right) \right]. \tag{3}$$

How does Δ and $\bar{\Delta}$ transform under gauge transformations? Is it an observable?

b) Now, we fix $\Delta_0 = |\Delta|$, and integrate out the fermions ψ, to obtain our effective action as, **2**pt(s)

$$S_{EM}[\theta, \boldsymbol{A}, \phi] := -\mathrm{tr} \ln \left(\mathcal{G}^{-1} \left[\Delta = \Delta_0 e^{2i\theta}, \bar{\Delta} = \Delta_0 e^{-2i\theta} \right] \right), \tag{4}$$

where

$$\hat{\mathcal{G}}^{-1} = \begin{pmatrix} \partial_\tau + i\phi + \frac{1}{2m}(-i\nabla - \boldsymbol{A})^2 - \mu & \Delta_0 e^{2i\theta} \\ \Delta_0 e^{-2i\theta} & \partial_\tau - i\phi - \frac{1}{2m}(-i\nabla + \boldsymbol{A})^2 + \mu \end{pmatrix} \tag{5}$$

is the Gor'kov's Greens's function (we also set $e = 1$). Since S_{EM} is invariant under gauge transformations, first gauge away the phase dependence of the order parameter field, by using the unitary transformation $\hat{U} \equiv \mathrm{diag}\left(e^{-i\theta}, e^{i\theta}\right)$ and then using $\theta = 0, \phi' = \phi + \partial_\tau \theta, \boldsymbol{A} + e^{-1}\boldsymbol{\nabla}\theta$ show that

$$\mathcal{G}^{-1} = U\mathcal{G}^{-1}U^\dagger = \underbrace{\tau_0 \partial_\tau + \tau_3 \left(-\frac{1}{2m}\nabla^2 - \mu \right) + \tau_1 \Delta_0}_{\hat{\mathcal{G}}_0^{-1}} + \underbrace{i\tau_3 \phi' + \frac{i}{2m}\tau_0 \{\nabla, \boldsymbol{A}'\}}_{\hat{\mathcal{X}}_1} + \underbrace{\tau_3 \frac{1}{2m}\boldsymbol{A}'^2}_{\hat{\mathcal{X}}_2}, \tag{6}$$

where τ_js are the Pauli matrices in Nambu space.

c) Now expand the tr ln and express the effective action as 2pt(s)

$$S_{EM} \sim \text{const.} - \operatorname{tr}\left(\mathcal{G}_0\mathcal{X}_1\right) - \operatorname{tr}\left(\mathcal{G}_0\mathcal{X}_2\right) + \frac{1}{2}\operatorname{tr}\left(\mathcal{G}_0\mathcal{X}_1\mathcal{G}_0\mathcal{X}_1\right). \tag{7}$$

Now given that the first order term $\operatorname{tr}(\mathcal{G}_0\mathcal{X}_1) \propto in \int d\tau d^d r \phi(\tau, \mathbf{r})$, is zero, show that the second order term $\propto \mathcal{X}_2$ is given as

$$-\operatorname{tr}\left(\mathcal{G}_0\mathcal{X}_2\right) = \frac{n}{2m}\int d\tau \int d^d\boldsymbol{r}\,\boldsymbol{A}^2(\tau,\boldsymbol{r}), \tag{8}$$

where n is the density of electrons.

d) Finally, show that 3pt(s)

$$\frac{1}{2}\operatorname{tr}\left(\mathcal{G}_0\mathcal{X}_1\mathcal{G}_0\mathcal{X}_1\right) = C_1\sum_q \phi'_q\phi'_{-q} + C_2\sum_q \boldsymbol{A}'_q\cdot\boldsymbol{A}'_q, \tag{9}$$

where

$$C_1 = \sum_p \frac{\omega_n^2 - E_{\boldsymbol{p}}^2 + 2\Delta_0^2}{\left(\omega_n^2 + E_{\boldsymbol{p}}^2\right)^2}, \quad C_2 = \sum_p \frac{1}{dm^2}\frac{\boldsymbol{p}^2\left(-\omega_n^2 + E_{\boldsymbol{p}}^2\right)}{\left(\omega_n^2 + E_{\boldsymbol{p}}^2\right)^2}. \tag{10}$$

Problem 8.2: Coefficients of the effective Lagrangian [**Oral** | 5 pt(s)]

ID: ex_em_super2:qft24

Learning objective

We will analyze the obtained effective action, namely the coefficients entering the Lagrangian and their temperature dependence.

In the previous task, we have obtained the effective action describing coupling between the phase of the superconducting order parameter and the electromagnetic field,

$$S_{\mathrm{EM}} = \int d\tau \int d^d\boldsymbol{r}\left[\mathcal{C}_1\left(\partial_\tau\theta + \phi\right)^2 + \left(\frac{n}{2m} + \mathcal{C}_2\right)\left(\nabla\theta - \boldsymbol{A}\right)^2\right], \tag{11}$$

where

$$\mathcal{C}_1 = \frac{T}{L^d}\sum_p \frac{\omega_n^2 - E_{\boldsymbol{p}}^2 + 2\Delta_0^2}{\left(\omega_n^2 + E_{\boldsymbol{p}}^2\right)^2}, \quad \mathcal{C}_2 = \frac{T}{L^d}\sum_p \frac{1}{dm^2}\frac{\boldsymbol{p}^2\left(-\omega_n^2 + E_{\boldsymbol{p}}^2\right)}{\left(\omega_n^2 + E_{\boldsymbol{p}}^2\right)^2}. \tag{12}$$

Here we will focus on calculating the coefficients $\mathcal{C}_1, \mathcal{C}_2$.

a) In the expressions for $\mathcal{C}_1, \mathcal{C}_2$ take the summation over fermionic Matsubara frequencies. 2pt(s)

b) Consider two limits, $\Delta_0/T \gg 1$ and $\Delta_0/T \ll 1$ ($\Delta_0 \neq 0$), to perform explicit summation over momentum. Try to come up with the physical meaning of the obtained coefficients and their temperature dependence. 3pt(s)

Hint: you may find it useful to assume that $T/\mu \ll 1$ and chemical potential lies far away from the band edges, so that $\xi_{\mathbf{p}} = \mathbf{p}^2/(2m) - \mu \approx v_{\mathrm{F}}(p - p_{\mathrm{F}})$.