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Problem 7.1: Cooper instability as a two-particle problem

[**Written** | 12 pt(s)]

ID: ex_cooper_pairs:qft24

> **Learning objective**
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> The Cooper instability is a fundamental mechanism behind the transition from the normal to the superconducting phase. In 1956, Cooper studied how it can be energetically favorable for two electrons to pair up in states above the Fermi surface rather than occupying states at the Fermi level. This destabilizes the Fermi surface and thereby initiates a phase transition to a new, superconducting, ground state.
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> In the lecture as well as in the next question you will treat the Cooper instability more formally within the framework of many body physics. However, to understand how an attractive electron-electron interaction can destabilize the Fermi-surface we will first treat the problem from a two-particle perspective.

We study a two-body problem with electrons, $j \in \{1,2\}$, at positions $\boldsymbol{r}_j \in \mathbb{R}^d$ with spins $\sigma_j \in \{\uparrow,\downarrow\}$. The corresponding Schrödinger equation is given by

$$\left[-\frac{1}{2m}\left(\boldsymbol{\nabla}^2_{\boldsymbol{r}_1} + \boldsymbol{\nabla}^2_{\boldsymbol{r}_2}\right) + V\left(\boldsymbol{r}_1 - \boldsymbol{r}_2\right)\right]\Psi_{\sigma_1,\sigma_2}\left(\boldsymbol{r}_1,\boldsymbol{r}_2\right) = E\Psi_{\sigma_1,\sigma_2}\left(\boldsymbol{r}_1,\boldsymbol{r}_2\right). \tag{1}$$

a) As a first step, transform Eq. (1) into relative, $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$, and center-of-mass, $\boldsymbol{R} = (\boldsymbol{r}_1 + \boldsymbol{r}_2)/2$, coordinates. Use $\Psi_{\sigma_1,\sigma_2}\left(\boldsymbol{r}_1,\boldsymbol{r}_2\right) = \widetilde{\Psi}_{\sigma_1,\sigma_2}(\boldsymbol{r},\boldsymbol{R})$ **2**pt(s)

b) Go to Fourier space, $\widetilde{\Psi}_{\sigma_1,\sigma_2}(\boldsymbol{r},\boldsymbol{R}) = \sum_{\boldsymbol{k},\boldsymbol{K}} e^{i\boldsymbol{K}\cdot\boldsymbol{R}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \phi_{\sigma_1,\sigma_2}(\boldsymbol{k},\boldsymbol{K})$, to bring Eq. (1) to the form **2**pt(s)

$$\left(2\epsilon_{\boldsymbol{k}} + E_{\boldsymbol{K}}\right)\phi_{\sigma_1,\sigma_2}(\boldsymbol{k},\boldsymbol{K}) + \sum_{\boldsymbol{k}'} V_{\boldsymbol{k}-\boldsymbol{k}'}\phi_{\sigma_1,\sigma_2}\left(\boldsymbol{k}',\boldsymbol{K}\right) = E\phi_{\sigma_1,\sigma_2}(\boldsymbol{k},\boldsymbol{K}), \tag{2}$$

where $V_{\boldsymbol{q}}$ is the Fourier transform of $V(\boldsymbol{r})$. Show that the state with lowest energy has $\boldsymbol{K} = 0$. What does this mean in terms of the momenta of the two individual electrons? From now on, we will only keep the $\boldsymbol{K} = 0$ component.

c) To connect better to the notation that will be introduced in the lecture, let us define $\Delta_{\sigma_1,\sigma_2}(\boldsymbol{k}) := \left(E - 2\epsilon_{\boldsymbol{k}}\right)\phi_{\sigma_1,\sigma_2}(\boldsymbol{k},\boldsymbol{0})$. Rewrite the Schrödinger equation in Eq. (2) in terms of $\Delta_{\sigma_1,\sigma_2}(\boldsymbol{k})$. As is common, we separate $\Delta_{\sigma_1,\sigma_2}(\boldsymbol{k})$ into the singlet component, $\psi_0(\boldsymbol{k})$, and the three triplet components, $\boldsymbol{d}(\boldsymbol{k}) = \left(d_1(\boldsymbol{k}), d_2(\boldsymbol{k}), d_3(\boldsymbol{k})\right)^T$ according to **2**pt(s)

$$\Delta_{\sigma_1,\sigma_2}(\boldsymbol{k}) = \psi_0(\boldsymbol{k})\left(i\hat{\sigma}_y\right)_{\sigma_1,\sigma_2} + \sum_{j=1}^{3} d_j(\boldsymbol{k})\left(i\hat{\sigma}_y\hat{\sigma}_j\right)_{\sigma_1,\sigma_2}, \tag{3}$$

where $\hat{\sigma}_j, j = 0,1,2,3$, are Pauli matrices. For electrons, the components must fulfill

$$\psi_0(\boldsymbol{k}) = \psi_0(-\boldsymbol{k}), \quad \boldsymbol{d}(\boldsymbol{k}) = -\boldsymbol{d}(-\boldsymbol{k}). \tag{4}$$

Why?

d) By plugging Eq. (3) into the Schrödinger equation that you previously expressed in terms of Δ, show that it decays into four separate equations for singlet and triplet components. We now simplify by focusing on **2pt(s)**

$$V(\boldsymbol{k}-\boldsymbol{k}') = \begin{cases} -V_0, & \text{if } |\epsilon_{\boldsymbol{k}} - E_F| < \Lambda \text{ and } |\epsilon_{\boldsymbol{k}'} - E_F| < \Lambda, \\ 0, & \text{otherwise }, \end{cases} \tag{5}$$

with $V_0 > 0$ (purely attractive interaction). Here E_F is the Fermi energy and Λ a cutoff (physically related to the Debye energy, see lecture). Convince yourself that your equations for $\boldsymbol{d}$ imply $\boldsymbol{d} = 0$ due to Eq. (4).

e) For the remaining singlet component, let us take **2pt(s)**

$$\psi_0(\boldsymbol{k}) = \begin{cases} \psi_0, & \text{if } |\epsilon_{\boldsymbol{k}} - E_F| < \Lambda \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

By introducing the density of states $\rho(\epsilon)$ (per spin) rewrite the equation for singlet pairing as

$$\psi_0 = V_0 \psi_0 \int_{E_F-\Lambda}^{E_F+\Lambda} \mathrm{d}\epsilon \frac{\rho(\epsilon)}{2\epsilon - E}. \tag{7}$$

f) Finally we are ready to study the physical consequences of the calculations above. Assume that there are many other additional particles in the Fermi sea and that $\Lambda \ll E_F$. We, thus, can approximate the (effective) density of states in Eq. (7) as $\rho(\epsilon) \sim \rho_0 \Theta(\epsilon - E_F)$. Note that the Θ-function takes care of the fact that the states below E_F are occupied by other electrons already and, hence, unavailable. Compute the binding energy $\delta E = E - 2E_F$ and show that there are states with $\delta E < 0$ for arbitrarily small attractive interactions $V_0 > 0$. What does δE correspond to and what are the physical consequences of $\delta E < 0$? **2pt(s)**

Problem 7.2: Cooper Instability from perturbation theory

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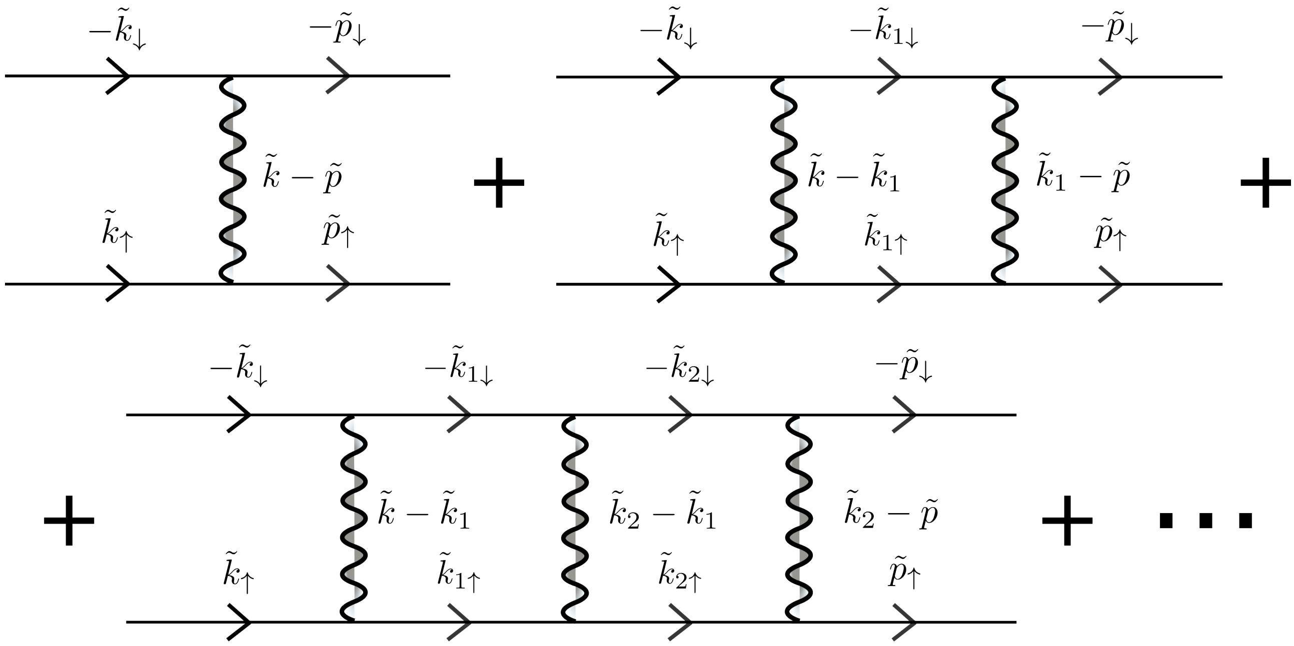
ID: ex_cooper_instability:qft24

Learning objective

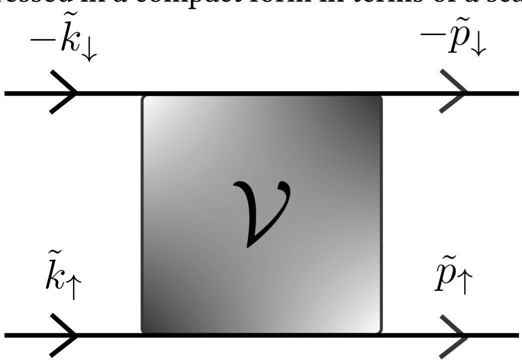
In this exercise we will use perturbation theory alongside Matsubara summation to calculate the scattering vertex amplitude of Cooper pairs. By investigating the corresponding instability of this object, we can extract the critical temperature T_C up to which superconductivity is defined.

We start by considering electrons, with opposite mommentum and spin, scattering once or multiple times through an effective interaction $V_{eff}(\mathbf{q}, i\omega)$ close to the Fermi Surface. These scatterings can be expressed via the following diagrams, with the four-momentum notation $\tilde{k} = (\boldsymbol{k}, i\omega_n)$,



This series is already expressed in terms of the dominant diagrams contributing to the formation of Cooper Pairs (*Ladder Diagrams*).

a) The series can also be expressed in a compact form in terms of a scattering vertex amplitude $\mathcal{V}$ as **2**pt(s)



Determine the form of $\mathcal{V}$ diagramatically and algebraically. For the later, you can adopt a point-like attractive interaction ($V > 0$) given by

$$V_{eff}(\mathbf{q}, i\omega) = \begin{cases} -V & \text{for } |i\omega_n| < \omega_D \\ 0 & \text{for } |i\omega_n| > \omega_D \end{cases}, \tag{8}$$

up to a cutoff given by the Debye energy ω_D. This implies that the electrons will interact in an energy range of $2\omega_D$ (with $\hbar = 1$) around the Fermi Surface.

b) Show that the denominator of the algebraic expression of $\mathcal{V}$ can be rewritten as **2**pt(s)

$$I = 1 - \frac{1}{2} V \rho\left(\varepsilon_{\mathrm{F}}\right) \int_0^{\omega_{\mathrm{D}}} \frac{\mathrm{d}\xi}{\xi} \tanh \frac{\xi}{2 k_{\mathrm{B}} T_{\mathrm{c}}}.$$

Identify the condition for the divergence of $\mathcal{V}$. What is the physical meaning of this instability?

Hint: For the first part of the exercise, remember that the density of states can be approximated to $\rho\left(\xi\right) \approx \rho\left(\varepsilon_F\right)$ in this case.

c) Finally, show that T_C is given explicitly by **2**pt(s)

$$T_{\mathrm{c}} = \frac{\mathrm{e}^{\gamma}}{\pi} 2\omega_{\mathrm{D}} \exp\left(-\frac{2}{V\rho\left(E_{\mathrm{F}}\right)}\right).$$

Hint: You may need the following integral

$$\int_0^\infty \frac{\ln x}{\cosh^2 x}\,\mathrm{d}x = -\ln\frac{4\mathrm{e}^{\gamma}}{\pi}, \tag{9}$$

where $\gamma = 0.5772...$ is the Euler-Mascheroni constant. Also notice that T_C for superconductors is typically small compared to the Debye temperature ω_D/k_B.