Prof. Dr. Mathias Scheurer
Institute for Theoretical Physics III, University of Stuttgart

May 15th, 2024
SS 2024

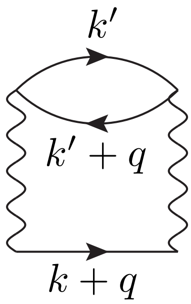
Problem 6.1: Self energy diagram in fermi liquid theroy

[**Written** | 10 pt(s)]

ID: ex_self_energy_fermi_liquid:qft24

> **Learning objective**
>
> In this exercise you will investigate one of the lowest-order diagrams that contributes to the imaginary part of the fermionic self-energy. This is particularly important in Fermi Liquid theory, for example, where the inverse of the imaginary part of the self-energy is related to the lifetime of quasiparticles.

In this exercise we will calculate the diagram



which plays a central role in the microscopic theory of Fermi liquids. Here the straight line represents the fermionic single-particle Green's function which we take to be of the form

$$G(i\omega_n, \boldsymbol{k}) = \frac{Z_{\boldsymbol{k}}}{i\omega_n - \epsilon_{\boldsymbol{k}}}, \qquad 0 < Z_{\boldsymbol{k}} \leq 1. \tag{1}$$

For simplicity, you can assume the dispersion to be parabolic, $\epsilon_{\boldsymbol{k}} = \boldsymbol{k}^2/(2m)$, although this is not necessary to solve the following problems. Furthermore, $k = (i\omega_n, \boldsymbol{k})$ and $q = (i\Omega_n, \boldsymbol{q})$ are used to comprise Matsubara frequencies and momenta with ω_n and Ω_n being fermionic and bosonic, respectively. The wiggly lines in the diagram refer to the four-fermion-interaction amplitude $U_{\boldsymbol{q}}$ that in general depends on the transferred momentum $\boldsymbol{q}$.

a) In order to understand the physical meaning of $Z_{\boldsymbol{k}}$, often referred to as "quasiparticle residue", calculate the spectral function and the occupation number of the single particle state $\boldsymbol{k}$ associated with $G(i\omega_n, \boldsymbol{k})$ in Eq. (1). **2**pt(s)

b) Returning to our ultimate goal of evaluating the diagram $\Sigma(i\omega_n, \boldsymbol{k})$ shown above, write down its analytical form (in Matsubara formalism) following from the Feynman rules discussed in the lecture. Identify the particle-hole bubble $\Pi(i\Omega_n, \boldsymbol{q})$ that has been a central building block in many calculations of the lecture course. **2**pt(s)

c) Using the residue theorem with a properly chosen integration contour in the complex plane and subsequent analytic continuation $i\omega_n \to \omega + i0^+$ to the real axis, show that the retarded form **2**pt(s)

$\Sigma^R(\omega, \boldsymbol{k})$ of the diagram is given by

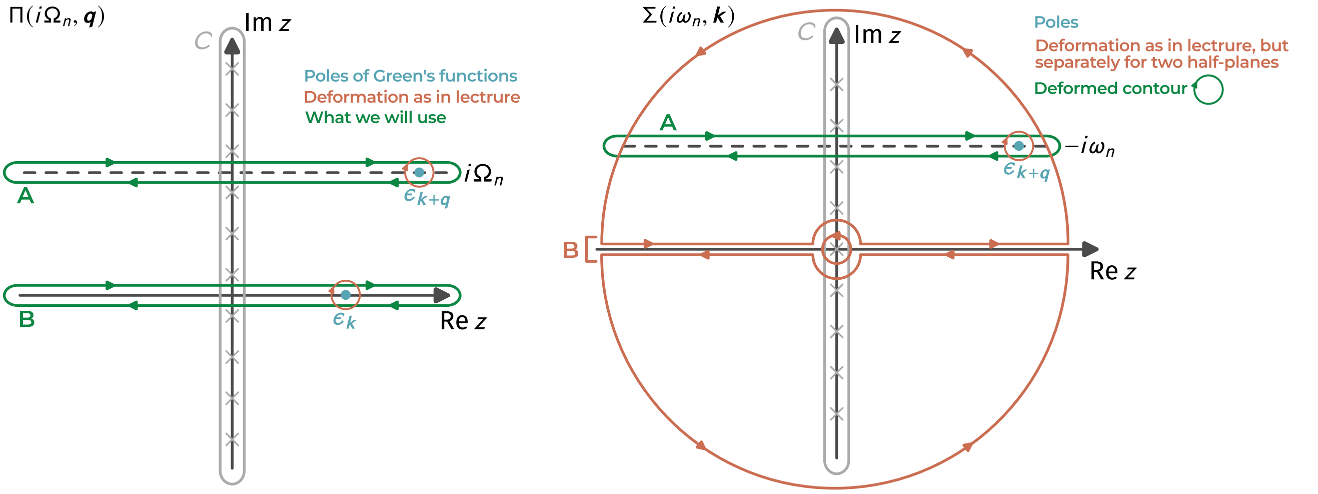
$$\Sigma^R(\omega, \boldsymbol{k}) = \int \frac{\mathrm{d}^d \boldsymbol{q}}{(2\pi)^d} U_{\boldsymbol{q}}^2 \left[\mathcal{P}\!\!\int \frac{\mathrm{d}\Omega}{2\pi} \coth\left(\frac{\Omega}{2T}\right) G^R(\omega+\Omega, \boldsymbol{k}+\boldsymbol{q})\, \mathrm{Im}\Pi^R(\Omega, \boldsymbol{q}) + \int \frac{\mathrm{d}\Omega}{2\pi} \tanh\left(\frac{\Omega+\omega}{2T}\right) \Pi^A(\Omega, \boldsymbol{q})\, \mathrm{Im}G^R(\omega+\Omega, \boldsymbol{k}+\boldsymbol{q}) \right] \tag{2}$$

with d denoting the dimensionality of the system, $\mathcal{P}\!\int$ the principle value integral, G^R/G^A the retarded/advanced Green's function, and $\Pi^R(\Omega, \boldsymbol{q})/\Pi^A(\Omega, \boldsymbol{q})$ the retarded/advanced particle-hole bubble determined by

$$\Pi^R(\Omega, \boldsymbol{q}) = 2 \int \frac{\mathrm{d}^d \boldsymbol{k}}{(2\pi)^d} \int \frac{\mathrm{d}\omega}{2\pi} \left[\tanh\left(\frac{\omega}{2T}\right) G^R(\omega+\Omega, \boldsymbol{k}+\boldsymbol{q})\, \mathrm{Im}G^R(\omega, \boldsymbol{k}) + \tanh\left(\frac{\omega+\Omega}{2T}\right) G^A(\omega, \boldsymbol{k})\, \mathrm{Im}G^R(\omega+\Omega, \boldsymbol{k}+\boldsymbol{q}) \right] \tag{3}$$

and similarly for $\Pi^A(\Omega, \boldsymbol{q})$.

Hint: Remember that 2T is related to the residue of the hyperbolic functions for $\mathrm{Res}\left[\coth(\beta z/2), i\Omega_n\right] = 2T$ and $\mathrm{Res}\left[\tanh(\beta z/2), i\Omega_n\right] = 2T$. You might also get some useful insights from the following figures.



d) Let us first focus on the imaginary part of $\Sigma^R(\omega, \boldsymbol{k})$. Convince yourself that $\Pi^R(\Omega, \boldsymbol{q})$ enters $\mathrm{Im}\Sigma^R(\omega, \boldsymbol{k})$ only in the form of its imaginary part $\mathrm{Im}\Pi^R(\Omega, \boldsymbol{q})$. Focusing on small T and ω (compared to the Fermi energy E_F), which allows neglecting ω and Ω in the delta functions appearing in the expression for $\mathrm{Im}\Pi^R(\Omega, \boldsymbol{q})$ following from Eq. (3), show that **2pt(s)**

$$\mathrm{Im}\Pi^R(\Omega, \boldsymbol{q}) \sim A_{\boldsymbol{q}}\, \Omega \tag{4}$$

and find the explicit form of the prefactor $A_{\boldsymbol{q}}$.

Hint: To obtain Eq. (4) you can take the density of states to be independent of the direction normal to the Fermi surface.

e) Using this result, obtain **2**pt(s)

$$\mathrm{Im}\Sigma^R(\omega, \boldsymbol{k}) \sim B_{\boldsymbol{k}} \left(\omega^2 + \pi^2 T^2\right) \tag{5}$$

for $\omega, T \ll E_F$. This is the typical behavior of a Fermi liquid. Determine an expression for the prefactor $B_{\boldsymbol{k}}$ in terms of the quasiparticle residues.