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### Problem 5.1: Polarization operator for graphene

[**Oral** | 4 pt(s)]

ID: ex_polarization_bubble_graphene:qft24

> **Learning objective**
>
> The random phase approximation (RPA) where the bare interaction gets renormalised to an effective interaction is ubiquitous in many-body systems leading to important effects such as screening. In this exercise, you will get familiar with the polarization operator which is the chief object required for calculating screening effects under RPA.

In this exercise we will calculate the retarded polarization operator $\Pi^R(\Omega, \boldsymbol{q})$ of a single Dirac cone of graphene, i.e., of a 2D system with Bloch Hamiltonian

$$h_{\boldsymbol{k}} = v_F(\sigma_x k_x + \sigma_y k_y). \tag{1}$$

Here $\sigma_x$ and $\sigma_y$ are Pauli matrices in pseudospin space.

a) Show that the Matsubara Green's function of the system can be written as **2**pt(s)

$$G_p = \sum_{\lambda=\pm} \frac{P_\lambda, \boldsymbol{p}}{-i\omega_n + \lambda v_F |\boldsymbol{p}|}, P_{\lambda,\boldsymbol{p}} = \frac{1}{2}\left(\sigma_0 + \lambda \boldsymbol{\sigma} \frac{\boldsymbol{p}}{|\boldsymbol{p}|}\right). \tag{2}$$

What is the meaning of the operators $\mathcal{P}_{\lambda,\boldsymbol{p}}$?

b) On the imaginary axis, the polarization operator is given by **2**pt(s)

$$\Pi(i\Omega_n, \boldsymbol{q}) = T \sum_{\omega_n} \int \frac{\mathrm{d}^2\boldsymbol{k}}{(2\pi)^2} \mathrm{Tr}\left[G(i\omega_n, \boldsymbol{k}) G(i\omega_n + i\Omega_n, \boldsymbol{k}+\boldsymbol{q})\right] \tag{3}$$

with $\Omega_n$ and $\omega_n$ being bosonic and fermionic, respectively. Evaluate the trace $\mathrm{Tr}[\dots]$, which refers to pseudospin space, using the representation (2) of the Green's function and the Matsubara summation to show that

$$\Pi\left(i\Omega_n, \boldsymbol{q}\right) = \frac{1}{\Omega} \sum_{\lambda,\lambda'} \sum_{\boldsymbol{p}} \frac{1}{2}\left[1 + \lambda\lambda' \cos(\sphericalangle(\boldsymbol{p}, \boldsymbol{p}+\boldsymbol{q}))\right] \times \frac{n_F\left(\epsilon_\lambda(\boldsymbol{p})\right) - n_F\left(\epsilon'_\lambda\left(\boldsymbol{p}+\boldsymbol{q}\right)\right)}{i\Omega_m + \epsilon_\lambda(\boldsymbol{p}) - \epsilon'_\lambda(\boldsymbol{p}+\boldsymbol{q})}. \tag{4}$$

### Problem 5.2: Polarization operator of graphene (continued)

[**Written** | 7 (+4 bonus) pt(s)]

ID: ex_polarization_bubble_graphene_2:qft24

> **Learning objective**
>
> This is a continuation of the previous exercise. We will now calculate the integration over the momentum for the polarization operator explicitly.

a) Identify the relevant terms in the expression for the polarization operator from the previous exercise and perform an analytic continuation $\Pi(i\Omega_n, \boldsymbol{q}) \to \Pi^R(\Omega, \boldsymbol{q})$ to show that **2**pt(s)

$$\Pi^R(\Omega, \boldsymbol{q}) = +\int \frac{d^2\boldsymbol{p}}{(2\pi)^2}(1 - \cos(\sphericalangle(\boldsymbol{p}, \boldsymbol{p}+\boldsymbol{q}))) \times \frac{v_F(|\boldsymbol{p}| + |\boldsymbol{p}+\boldsymbol{q}|)}{(\Omega + i0^+)^2 - [v_F(|\boldsymbol{p}| + |\boldsymbol{p}+\boldsymbol{q}|)]^2}. \quad (5)$$

**Hint:** Remember that at $T = 0$

$$n_F(\epsilon_\lambda(\boldsymbol{p})) = \begin{cases} 0 & \lambda = + \\ 1 & \lambda = - \end{cases} \quad (6)$$

and that an analytic continuation can be performed with $i\Omega_n \to \Omega + i0^+$.

b) To perform the momentum integration, it is convenient to introduce new variables $\xi$ and $\eta$ via **3**pt(s)

$$\xi = |\boldsymbol{p}| + |\boldsymbol{p}+\boldsymbol{q}|, \qquad \eta = |\boldsymbol{p}| - |\boldsymbol{p}+\boldsymbol{q}|. \quad (7)$$

What is the geometric meaning of $\xi$ and $\eta$? In order to rewrite the momentum integration, show that

$$\cos(\theta_{\boldsymbol{p},\boldsymbol{p}+\boldsymbol{q}}) = \frac{2\boldsymbol{q}^2 - \eta^2 - \xi^2}{\eta^2 - \xi^2} \quad (8)$$

with $\theta_{\boldsymbol{p},\boldsymbol{p}+\boldsymbol{q}}$ denoting the angle between $\boldsymbol{p}$ and $\boldsymbol{p}+\boldsymbol{q}$ and that

$$\mathrm{d}p_x \mathrm{d}p_y = \frac{\xi^2 - \eta^2}{4\sqrt{(\boldsymbol{q}^2 - \eta^2)(\xi^2 - \boldsymbol{q}^2)}} \mathrm{d}\xi \mathrm{d}\eta. \quad (9)$$

You might find it useful to choose $\boldsymbol{q} = (q_x, 0)$ which is possible without loss of generality.

c) Rewrite the momentum integral in terms of $\xi$ and $\eta$ and perform the integrations using the identity **2**pt(s)

$$\int_1^\infty \mathrm{d}x \frac{x}{\sqrt{x^2 - 1}} \frac{1}{w^2 - x^2} = \frac{-\pi}{2\sqrt{1 - w^2}} \quad (10)$$

for $w \in \mathbb{C}$ (except for $w \in \mathbb{R}$ with $|w| \geq 1$).

*d) Sketch $\mathrm{Re}\Pi^R(\Omega, \boldsymbol{q})$ and $\mathrm{Im}\Pi^R(\Omega, \boldsymbol{q})$ as a function of $\Omega$. **+2**pt(s)

*e) Check your result for $\mathrm{Im}\Pi^R(\Omega, \boldsymbol{q})$ by using $\frac{1}{(x+i0^+)} = \mathcal{P}\frac{1}{x} - i\pi\delta(x)$ before performing the $\xi$ integration. **+2**pt(s)