Prof. Dr. Mathias Scheurer
Institute for Theoretical Physics III, University of Stuttgart

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Problem 12.1: Feynman parameters

[**Written** | 5 pt(s)]

ID: ex_feynman_parameters:qft24

> **Learning objective**
>
> The purpose of this problem is to familiarize with the concept of Feynman parameters. It is for example used for the calculation of integrals appearing in loop diagrams and introduces additional parameters in order to bring the integrals into a form more suitable for calculation.

a) Begin with the simple case of two factors in the denominator: **1**pt(s)

$$\frac{1}{AB} = \int_0^1 dx\, \frac{1}{[xA + (1-x)B]^2} = \int_0^1 dx\, dy\, \delta(x+y-1)\, \frac{1}{[xA+yB]^2} \,. \tag{1}$$

By differentiating with respect to B, prove that

$$\frac{1}{AB^n} = \int_0^1 dx\, dy\, \delta(x+y-1)\, \frac{ny^{n-1}}{[xA+yB]^{n+1}} \,. \tag{2}$$

b) Use Eq. (2) and show by induction that **2**pt(s)

$$\frac{1}{A_1 A_2 \cdots A_n} = \int_0^1 dx_1\ dx_2 \cdots dx_n\, \delta\Big(\sum x_i - 1\Big)\, \frac{(n-1)!}{[x_1 A_1 + x_2 A_2 + \cdots x_n A_n]^n} \,. \tag{3}$$

c) Finally, prove **2**pt(s)

$$\frac{1}{A_1^{m_1} A_2^{m_2} \cdots A_n^{m_n}} = \int_0^1 dx_1\ dx_2\ \cdots dx_n\, \delta\Big(\sum x_i - 1\Big)\, \frac{\prod x_i^{m_i - 1}}{[\sum x_i A_i]^{\sum m_i}}\, \frac{\Gamma(m_1 + \cdots + m_n)}{\Gamma(m_1)\cdots\Gamma(m_n)} \,, \tag{4}$$

with $\Gamma(x)$ the gamma function and $\Gamma(n) = (n-1)!$ for positive integer n.

Problem 12.2: The electron self-energy

[**Oral** | 9 pt(s)]

ID: ex_electron_self_energy:qft24

> **Learning objective**
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> The mass-energy equivalence inherent to any relativistic theory implies for quantum field theories that fluctuations of fields around particles with "bare" mass m_0 shift the latter to a larger, observable mass m. In QED, virtual photons that couple to the charged electron make up for its *self-energy* which, in turn, contributes to its mass m; we say that the mass is *renormalized*. As a result, we find that m_0 and m differ

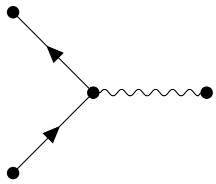
by an infinity.

In quantum electrodynamics (QED) the Feynman rules read as

$$= \frac{i}{\not{p} - m_0 + i0^+} = \frac{i(\not{p} + m_0)}{p^2 - m_0^2 + i0^+}\,, \tag{5}$$

$$= \frac{-i\eta_{\mu\nu}}{q^2 + i0^+} \tag{6}$$

and

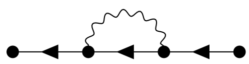


$$= -ie\gamma^\mu \tag{7}$$

for the electron and photon propagators as well as vertices, respectively.

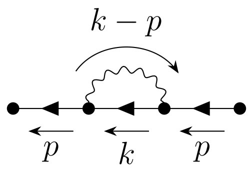
The dressed electron propagator is given by the sum of diagrams

$+$



$$+ \ldots \tag{8}$$

where the first diagram is just the free-field propagator, given in Eq. (5) and the second diagram (the *electron self-energy*) yields the expression



$$= \frac{i(\not{p} + m_0)}{p^2 - m_0^2}[-i\Sigma(p)]\frac{i(\not{p} + m_0)}{p^2 - m_0^2}. \tag{9}$$

a) Show Eq. (5), i.e. **1**pt(s)

$$\frac{i}{\not{p} - m_0 + i0^+} = \frac{i(\not{p} + m_0)}{p^2 - m_0^2 + i0^+} \tag{10}$$

by inverting the 4×4 matrix $\not{p}$. Use this as well as the Feynman rules to show that the self-energy is given by

$$-i\Sigma(p) = (-ie)^2 \int \frac{d^4k}{(2\pi)^4}\,\gamma^\mu\,\frac{i(\not{k} + m_0)}{k^2 - m_0^2 + i0^+}\,\gamma_\mu\,\frac{-i}{(p-k)^2 - \mu^2 + i0^+}, \tag{11}$$

where m_0 is the bare mass of the electron and $\mu > 0$ is a small photon mass to regulate the infrared divergence of the integral.

b) Using Feynman parameters, show that the self-energy $-i\Sigma(p)$ takes the form **3**pt(s)

$$-i\Sigma(p) = -e^2 \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4} \frac{-2x\not{p} + 4m_0}{[\ell^2 - \Delta_\mu + i0^+]^2}\,, \tag{12}$$

where $\ell = k - xp$ and $\Delta_\mu = -x(1-x)p^2 + x\mu^2 + (1-x)m_0^2$.
Hint: The γ matrices fulfill $\gamma^\mu\gamma_\mu = 4$ and $\gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu$.

c) To control the ultraviolet divergence of the integral (12), use the *Pauli-Villars regularization* **3**pt(s)

$$\frac{1}{(p-k)^2 - \mu^2 + i0^+} \to \frac{1}{(p-k)^2 - \mu^2 + i0^+} - \frac{1}{(p-k)^2 - \Lambda^2 + i0^+} \tag{13}$$

for $\Lambda \to \infty$ and show that

$$\Sigma(p) = \frac{\alpha}{2\pi} \int_0^1 dx\,(2m_0 - x\not{p})\,\log\left[\frac{x\Lambda^2}{(1-x)m_0^2 + x\mu^2 - x(1-x)p^2}\right] \tag{14}$$

in this limit. The fine-structure constant α (in natural units) is given by $\alpha = e^2/4\pi$.
Hint: By a Wick rotation $\ell_E^0 = -i\ell^0$ integrals of the form $\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{(\ell^2-\Delta+i0^+)^2}$ can be transformed into 4 dimensional spherical coordinates. Evaluating the angular part yields

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{(\ell^2 - \Delta + i0^+)^2} = i\int_0^\infty \frac{d\ell_E}{8\pi^2} \frac{\ell_E^3}{(\ell_E^2 + \Delta)^2}. \tag{15}$$

d) Using the expression for the self-energy obtained in b), calculate the mass shift **2**pt(s)

$$\delta m = m - m_0 = \Sigma(\not{p} = m) \approx \Sigma(\not{p} = m_0) \tag{16}$$

in first order of α.

Show that the bare mass m_0 and the measurable mass m differ by a diverging quantity.