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Problem 11.1: Magnon-Magnon interactions

[**Oral** | 7 (+1 bonus) pt(s)]

ID: ex_holstein_magnon_self_energy_2:qft24

> **Learning objective**
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> As a continuation of the previous sheet, we will investigate the corrections on the spectrum of spin waves due to magnon-magnon interactions and check that to first order in perturbation theory, there is a correction to the bare dispersion $\omega_{\boldsymbol{k}}$ proportional to $T^{5/2}$ confirming that, indeed for small T, these interactions will not be dominant.

For finite temperatures, magnon-magnon interaction effects will become more pronounced and produce corrections to the spin-wave dispersion $\omega_{\boldsymbol{k}}$ obtained previously. Within second-order perturbation theory, the real part of the renormalized spectrum reads as

$$\omega_{\boldsymbol{k}}^{(2)} = \omega_{\boldsymbol{k}} + \Sigma^{(1)}(\boldsymbol{k}) + \dots \tag{1}$$

Our goal will be to explicitly calculate the self-energy correction $\Sigma^{(1)}(\boldsymbol{k})$.

a) First, show that the quartic interaction term can be rewritten as **2pt(s)**

$$\mathcal{H}_4^{HP} = -\frac{zJ}{N} \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4} \delta_{\boldsymbol{k}_1+\boldsymbol{k}_2,\boldsymbol{k}_3+\boldsymbol{k}_4} v^{HP}_{\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4} a^\dagger_{\boldsymbol{k}_1} a^\dagger_{\boldsymbol{k}_2} a_{\boldsymbol{k}_3} a_{\boldsymbol{k}_4} \tag{2}$$

with

$$v^{HP}_{\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4} = \frac{1}{2}\gamma_{\boldsymbol{k}_1-\boldsymbol{k}_3} - S\left(1-\sqrt{1-\frac{1}{2S}}\right)(\gamma_{\boldsymbol{k}_1}+\gamma_{\boldsymbol{k}_2}) + \text{ c.c. }. \tag{3}$$

Note that this potential is invariant under the exchange $\boldsymbol{k}_1 \leftrightarrow \boldsymbol{k}_2$ and $\boldsymbol{k}_3 \leftrightarrow \boldsymbol{k}_4$ such that it can be rewritten as

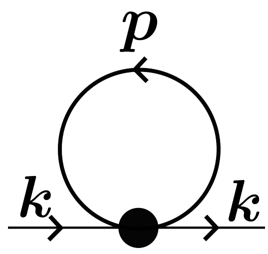
$$\begin{aligned} v^{HP}_{\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4} =& \frac{1}{4}\left(\gamma_{\boldsymbol{k}_1-\boldsymbol{k}_3}+\gamma_{\boldsymbol{k}_1-\boldsymbol{k}_4}+\gamma_{\boldsymbol{k}_2-\boldsymbol{k}_3}+\gamma_{\boldsymbol{k}_2-\boldsymbol{k}_4}\right) - S\left(1-\sqrt{1-\frac{1}{2S}}\right) \\ & \times \left(\gamma_{\boldsymbol{k}_1}+\gamma_{\boldsymbol{k}_2}+\gamma_{\boldsymbol{k}_3}+\gamma_{\boldsymbol{k}_4}\right). \end{aligned} \tag{4}$$

*b) Show that (3) and (4) are equivalent, according to the previous explanation. **+1pt(s)**

c) The dressed Green's function upto first order can be written as **3pt(s)**

$$G^{(1)}_{\boldsymbol{k}}(\omega) = G^{(0)}_{\boldsymbol{k}}(\omega) + \Sigma^{(1)}(\boldsymbol{k})\left[G^{(0)}_{\boldsymbol{k}}(\omega)\right]^2 \tag{5}$$

and is given by the diagram below



The multiplicity factor (4) accounts for the different number of equivalent diagrams. This can be found after applying Wick's theorem and keeping track only of connected diagrams in (2) after changing the basis to field operators. The contractions imply that $\boldsymbol{k}_1 = \boldsymbol{k}_4 = \boldsymbol{k}$ and $\boldsymbol{k}_3 = \boldsymbol{k}_2 = \boldsymbol{p}$.

Show that the self-energy contribution associated with this diagram (right-side) and to order $1/S$ is given by

$$\Sigma^{(1)}(\boldsymbol{k}) = -\frac{2zJ}{N}\sum_{\boldsymbol{p}} (1 + \gamma_{\boldsymbol{k}-\boldsymbol{p}} - \gamma_{\boldsymbol{k}} - \gamma_{\boldsymbol{p}})\, n_{\mathrm{B}}(\omega_{\boldsymbol{p}}) = -\alpha\frac{\omega_{\boldsymbol{k}}}{S} \tag{6}$$

where

$$\alpha = \frac{1}{N}\sum_{\boldsymbol{p}} (1 - \gamma_{\boldsymbol{p}})\, n^{(0)}_{\boldsymbol{p}}, \tag{7}$$

and $n_{\mathrm{B}}(\omega_{\boldsymbol{p}}) = 1/\left(e^{\omega_{\boldsymbol{p}}/T} - 1\right)$ is the Bose-Einstein distribution.

Hint: Use the fact that $\sum_{\boldsymbol{p}} \gamma_{\boldsymbol{k}-\boldsymbol{p}} n_{\mathrm{B}}(\omega_{\boldsymbol{p}}) = \sum_{\boldsymbol{p}} \gamma_{\boldsymbol{k}}\gamma_{\boldsymbol{p}} n_{\mathrm{B}}(\omega_{\boldsymbol{p}})$ and the dispersion obtained in question d) of problem Set 10 (eq. 16) to get the second equality in (6).

d) Finally, from eq. (6), show that magnon interactions produce a $T^{5/2}$ correction to the bare spin wave dispersion given by **2pt(s)**

$$\Sigma^{(1)}(\boldsymbol{k}) \propto k^2 \int \frac{d^3p}{(2\pi)^3}\frac{p^2}{e^{Ap^2/T} - 1} \propto k^2 \left(\frac{T}{A}\right)^{5/2}. \tag{8}$$

Hint: You may find the following integral useful

$$\int_0^\infty dx \frac{x^{\frac{3}{2}}}{e^x - 1} = \frac{3}{4}\sqrt{\pi}\zeta\left(\frac{5}{2}\right). \tag{9}$$

Problem 11.2: Free-particle solutions of the Dirac equation [**Written** | 5 (+2 bonus) pt(s)]

ID: ex_free_particle_solutions_dirac_equation:qft24

Learning objective

In the lecture you will start with high energy physics soon. This exercise will be a refresher on relativistic quantum mechanics and you will calculate different properties of solutions of the free Dirac equation.

The Dirac equation is given by

$$(i\gamma^\mu \partial_\mu - m)\,\psi = 0 \tag{10}$$

where the γ matrices in the Weyl representation are $\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$ and $\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$. Here σ^i are the standard Pauli matrices. As you might know from previous courses, the general solution of the Dirac equation can be written as a superposition of plane waves whose positive-frequency solutions are given by

$$\psi(x) = u(p)e^{-ip\cdot x}\,, \qquad p^2 = m^2\,, \qquad p^0 > 0\,. \tag{11}$$

$p_\mu = (p^0, \boldsymbol{p})$ and $x_\mu = (t, \boldsymbol{x})$ are 4-vectors and their inner product is defined as $p \cdot x = p^\mu x_\mu = p_\mu \eta^{\mu\nu} x_\nu = p^0 t - \boldsymbol{p} \cdot \boldsymbol{x}$. For the Minkowski metric η we use the $(+ - - -)$ convention.

a) Show that $u(p)$ must fulfill **1**pt(s)

$$(\gamma^\mu p_\mu - m)u(p) = 0. \tag{12}$$

Write the γ matrices in terms of Pauli matrices to bring the equation in a clearer form.

We denote the two independent solutions for $u(p)$

$$u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\xi^s \end{pmatrix}\,, \qquad s = 1,2 \tag{13}$$

with the two-component spinors $\xi^s \in \mathbb{C}^2$ which are normalized according to $(\xi^s)^\dagger \xi^r = \delta^{rs}$. Here, $\sigma^\mu = (1, \boldsymbol{\sigma})$ and $\bar{\sigma}^\mu = (1, -\boldsymbol{\sigma})$ with the Pauli vector $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)^T$.

Note: The operators $p\cdot\sigma$ and $p\cdot\bar{\sigma}$ have a positive spectrum and thus their square roots can be uniquely defined via $\sqrt{p\cdot\sigma}^2 = p\cdot\sigma$ and $\sqrt{p\cdot\bar{\sigma}}^2 = p\cdot\bar{\sigma}$, respectively.

b) To show that $u^s(p)$ are solutions of Eq. (12) first prove the the identity $(p\cdot\sigma)(p\cdot\bar{\sigma}) = p^2 = m^2$. **1**pt(s)

c) Use b) to show that $u^s(p)$ solves Eq. (12) for any ξ^s. **1**pt(s)

d) Show that $u^{r\dagger}(p)u^s(p)$ is not Lorentz invariant. Show that instead $u(p)$ can be normalized in a Lorentz invariant way according to **2**pt(s)

$$\bar{u}^r(p)u^s(p) = 2m\delta^{rs} \tag{14}$$

with $\bar{u} = u^\dagger\gamma^0$.

*e) Similarly to above, the negative-frequency solutions **+2**pt(s)

$$\psi(x) = v(p)e^{ipx}\,, \qquad p^2 = m^2\,, \qquad p^0 > 0 \tag{15}$$

can be obtained with two linearly independent solutions

$$v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\eta^s \end{pmatrix}\,, \qquad s = 1,2\,, \tag{16}$$

where η^s is another basis of two-component spinors. These solutions are normalized according to $\bar{v}^r(p)v^s(p) = -2m\delta^{rs}$.

Show that $\bar{u}^r(p)v^s(p) = \bar{v}^r(p)u^s(p) = 0$ and $u^{r\dagger}(p)v^s(p) \neq 0$ as well as $v^{r\dagger}(p)u^s(p) \neq 0$. However, show that reversing the sign of the 3-momentum in one factor of each spinor product leads to $u^{r\dagger}(\boldsymbol{p})v^s(-\boldsymbol{p}) = v^{r\dagger}(\boldsymbol{p})u^s(-\boldsymbol{p}) = 0$.