Prof. Dr. Mathias Scheurer
Institute for Theoretical Physics III, University of Stuttgart

June 19th, 2024
SS 2024

Problem 10.1: Mapping the Ising model to ϕ^4 theroy

[**Written** | 5 pt(s)]

ID: ex_non_linear_sigma_model_expansion:qft24

Learning objective

In this exercise you will perform a gradient expansion and show that the effective action in the partition function of the Ising model can be mapped to the well studied ϕ^4 theory with an additional magnetic term.

In the lecture you derived the effective action after the Hubbard-Stratonovich transformation

$$S_{\text{eff}}[\phi] = \sum_{ij} \phi_i \bar{J}_{ij} \phi_j - \sum_i \phi_i \bar{h}_i - \sum_i \ln\cosh\left[2 \sum_j \bar{J}_{ij}\phi_j\right], \tag{1}$$

which will be the starting point of this exercise. We assume that the coupling $\bar{J}$ is takes the form $\bar{J}_{ij} = f(|i-j|)$ for some function f that does not vary over space rapidly. This allows us to express $\bar{J}_{ij}$ in Fourier space according to

$$\bar{J}_{ij} = \frac{1}{N} \sum_{\boldsymbol{k}} e^{-i\boldsymbol{k}\cdot(\boldsymbol{r}_i - \boldsymbol{r}_j)} J(\boldsymbol{k}) \tag{2}$$

and neglect higher orders of $J(\boldsymbol{k}) = J(0) + \frac{1}{2}J''(0)\boldsymbol{k}^2 + \mathcal{O}(\boldsymbol{k}^4) \approx J(0) + \frac{1}{2}J''(0)\boldsymbol{k}^2$. Why is there no linear term?

a) As a first step, use $\phi_i = \frac{1}{\sqrt{N}} \sum_{\boldsymbol{k}} e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_i}\phi(\boldsymbol{k})$ and Eq. (2) to rewrite Eq. (1) in Fourier space. Expand $J(\boldsymbol{k})$ up to second order in $\boldsymbol{k}$. **1pt(s)**

b) Expand the ln cosh term according to $\ln\cosh(x) \approx \frac{x^2}{2} - \frac{x^4}{12}$ and only keep the terms up to ϕ^4 and $\boldsymbol{k}^2\phi^2$. **2pt(s)**

c) Finally, show that the real space representation of the remaining terms is **2pt(s)**

$$S_{\text{eff}}[\phi] = \int d^d\boldsymbol{x} \left[\frac{c}{2}(\boldsymbol{\nabla}\phi)^2 + \frac{r}{2}\phi^2 + g\phi^4 + h\phi\right] \tag{3}$$

in the continuum limit. What are the explicit forms of of the prefactors c, r, g, and h?

Hint: You might stumble across terms of the form $\int d^d\boldsymbol{x}\phi(\boldsymbol{x})\left[\boldsymbol{\nabla}^2\phi(\boldsymbol{x})\right]$. They can be related to $\int d^d\boldsymbol{x}\left[\boldsymbol{\nabla}\phi(\boldsymbol{x})\right]^2$ via partial integration.

Problem 10.2: Holstein-Primakoff Transformation

[**Oral** | 6 (+3 bonus) pt(s)]

ID: ex_holstein_magnon_self_energy:qft24

Learning objective

Spin degrees of freedom can be represented in terms of different fermionic or bosonic second-quantized operators that still obey the SU(2) algebra. This approach can not only bring new insights between quantum and classical spin models, but in more drastic cases even provide exact solutions, like in the paradigmatic example of the Kitaev spin liquid with a Majorana representation.

In this exercise, you will see how a bosonic mapping of spins, called *Holstein-Primakoff transformation*, can be used to naturally describe the spectrum of excitations of magnetically ordered phases, the so-called *magnons*. This mapping also provides a natural step for using many-body peturbation theory, which we shall do to investigate the effect of magnon-magnon interactions on the spectrum of spin waves in the next sheet.

Consider the Heisenberg Hamiltonian given by

$$\mathcal{H} = -\sum_{\langle i,j\rangle} J_{i,j}\boldsymbol{S}_i \cdot \boldsymbol{S}_j = -J\sum_{i,\boldsymbol{\delta}} \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+\delta}, \tag{4}$$

where $\boldsymbol{\delta}$ indicate nearest-neighboring vectors. Generally, the coupling constants $J_{i,j}$ can describe anisotropies, but we will assume for simplicity the uniform ($J_{i,j} = J$) and ferromagnetic ($J > 0$) case for this exercise.

The spins in (4) can be mapped to bosonic creation ($a_i^\dagger$) and destruction (a_i) operators that satisfy the commutation rules

$$\left[a_i, a_j^\dagger\right] = \delta_{i,j}, \quad [a_i, a_j] = \left[a_i^\dagger, a_j^\dagger\right] = 0, \tag{5}$$

with the Holstein-Primakoff (HP) transformation, defined by

$$\begin{aligned} S_i^+ &= S_i^x + iS_i^y = \sqrt{2S}\left(1 - \frac{a_i^\dagger a_i}{2S}\right)^{\frac{1}{2}} a_i, \\ S_i^- &= S_i^x - iS_i^y = \sqrt{2S}a_i^\dagger\left(1 - \frac{a_i^\dagger a_i}{2S}\right)^{\frac{1}{2}}, \\ S_i^z &= S - a_i^\dagger a_i, \end{aligned} \tag{6}$$

such that $n_i \leq 2S$ in the bosonic subspace. The right side of $S_i^\pm$ can be expanded such that

$$\begin{aligned} S_i^+ =& \sqrt{2S}\left[1 - \frac{a_i^\dagger a_i}{4S} - \frac{\left(a_i^\dagger a_i\right)^2}{32S^2}\cdots\right] a_i = \sqrt{2S}\left\{1 - \left(1 - \sqrt{1 - \frac{1}{2S}}\right) a_i^\dagger a_i \right. \\ &\left. + \left[1 - \sqrt{1 - \frac{1}{2S}} - \frac{1}{2}\left(1 - \sqrt{1 - \frac{1}{S}}\right)\right]\left(a_i^\dagger\right)^2 (a_i)^2 \cdots\right\} a_i, \end{aligned} \tag{7}$$

and

$$\begin{aligned} S_i^- =&\sqrt{2S}a_i^\dagger\left[1-\frac{a_i^\dagger a_i}{4S}-\frac{\left(a_i^\dagger a_i\right)^2}{32S^2}\cdots\right] = \sqrt{2S}a_i^\dagger\left\{1-\left(1-\sqrt{1-\frac{1}{2S}}\right)a_i^\dagger a_i\right.\\ &\left.+\left[1-\sqrt{1-\frac{1}{2S}}-\frac{1}{2}\left(1-\sqrt{1-\frac{1}{S}}\right)\right]\left(a_i^\dagger\right)^2(a_i)^2\cdots\right\}. \end{aligned} \tag{8}$$

a) Show that the operators defined in (6) satisfy the SU(2) algebra, i.e., the following commutation relations **2pt(s)**

$$\left[S_i^z,S_j^\pm\right]=\pm S_i^\pm\delta_{i,j} \quad , \quad \left[S_i^+,S_j^-\right]=2S_i^z\delta_{i,j}. \tag{9}$$

Hint: Consider the expansions in eq. (7-8).

*b) Why is it necessary to demand $n_i=a_i^\dagger a_i\leq 2S$ to ensure that we have only physical states in the bosonic subspace? **+1pt(s)**

c) By using the HP transformation, the Heisenberg Hamiltonian (4) can now be expressed in a bosonic form as **2pt(s)**

$$\mathcal{H}^{HP}=E_0^{HP}+\mathcal{H}_2^{HP}+\mathcal{H}_4^{HP}+\mathcal{H}_6^{HP}+\ldots \tag{10}$$

Show that up to and including quartic bosonic interaction terms we obtain

$$E_0^{HP}=-zJS^2N, \tag{11}$$

$$\mathcal{H}_2^{HP}=JS\sum_{i,\boldsymbol{\delta}}\left(a_i^\dagger a_i+a_{i+\boldsymbol{\delta}}^\dagger a_{i+\boldsymbol{\delta}}-a_{i+\boldsymbol{\delta}}^\dagger a_i-a_i^\dagger a_{i+\boldsymbol{\delta}}\right), \tag{12}$$

and

$$\begin{aligned} \mathcal{H}_4^{HP}=&-J\sum_{i,\delta}\left[\frac{1}{2}\left(a_i^\dagger a_{i+\delta}^\dagger a_i a_{i+\boldsymbol{\delta}}+\text{ h.c. }\right)-S\left(1-\sqrt{1-\frac{1}{2S}}\right)\right.\\ &\left.\times\left(a_{i+\boldsymbol{\delta}}^\dagger a_i^\dagger a_i a_i+a_{i+\boldsymbol{\delta}}^\dagger a_{i+\boldsymbol{\delta}}^\dagger a_{i+\boldsymbol{\delta}}a_i+\text{ h.c. }\right)\right]. \end{aligned} \tag{13}$$

Notice that $S\left(1-\sqrt{1-\frac{1}{2S}}\right)\approx\frac{1}{4}\left(1+\frac{1}{8S}+\frac{1}{32S^2}\right)$ by expanding the first few terms of the series expansion in powers of $\frac{1}{S}$.

In eqs. (11-13), N represents the number of spins, z the coordination number, E_0^{HP} the ground-state energy of the ferromagnetic state, the bilinear interaction term $\mathcal{H}_2^{HP}$ describes non-interacting magnons, and $\mathcal{H}_4^{HP}$ the first term in expansion (10) that takes into account magnon-magnon interactions.

d) So far, we have only considered *localized* spin deviations with the operators a_i and $a_i^\dagger$. We define the *spin wave* or *magnon* creation and destruction operators using Fourier transforms of the localized operators **2pt(s)**

$$a_{\boldsymbol{k}}=\frac{1}{\sqrt{N}}\sum_i e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_i}a_i, \quad a_{\boldsymbol{k}}^\dagger=\frac{1}{\sqrt{N}}\sum_i e^{i\boldsymbol{k}\cdot\boldsymbol{r}_i}a_i^\dagger \tag{14}$$

which obey

$$\left[a_{\boldsymbol{k}}, a_{\boldsymbol{k}'}^{\dagger}\right] = \delta_{\boldsymbol{k},\boldsymbol{k}'}, \quad [a_{\boldsymbol{k}}, a_{\boldsymbol{k}'}] = \left[a_{\boldsymbol{k}}^{\dagger}, a_{\boldsymbol{k}'}^{\dagger}\right] = 0 \tag{15}$$

as a consequence of (5). Using the magnon operators, (i) show that

$$\mathcal{H}_2^{HP} = zJS \sum_{\boldsymbol{k}} \left[(1-\gamma_{\boldsymbol{k}}) + \text{ c.c. }\right] a_{\boldsymbol{k}}^{\dagger} a_{\boldsymbol{k}}. \tag{16}$$

Note: For all Bravais lattices, $\gamma_{\boldsymbol{k}} = \gamma_{-\boldsymbol{k}} = \gamma_{\boldsymbol{k}}^*$.

(ii) Show that the energy spectrum vanishes quadratically $\omega_k \to A\boldsymbol{k}^2$ as $\boldsymbol{k} \to 0$, where A is the spin wave stiffness.

Keeping only the term (16) in expansion (10) is known as a *Harmonic Approximation*. This Hamiltonian describes an ideal gas of non-interacting magnons (Bose gas), which is an accurate description for small temperatures T when the interactions between magnons can be neglected.

*e) Realistically, the low temperature behavior of compounds demands the inclusion of anisotropic terms to (4). For ferromagnets, one important example is the single-ion anisotropy given by **+2**pt(s)

$$\mathcal{H}_D = -D \sum_i \left(S_i^z\right)^2 \tag{17}$$

due to the effect of a crystal field. If $D > 0$ ($D < 0$) the spins are forced to point along the z 'easy-axis' (xy 'easy-plane'). How is the spin-wave dispersion $\omega_{\boldsymbol{k}}$ changed in the previous question with the addition of this term?