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Information on lecture and tutorials

Here a few infos on the modalities of the course ***"Quantum Field Theory"***:

- The C@MPUS-ID of this course is 045570002.
- You can find detailed information on lecture and tutorials on the website of our institute:

  `https://itp3.info/qft24`
- **Written** problems have to be handed in via ILIAS and will be corrected by the tutors. You must earn at least **50 %** of the written points to be admitted to the exam.
- **Oral** problems have to be prepared for the exercise session and will be presented by a student at the blackboard. You must earn at least **66 %** of the oral points to be admitted to the exam.
- Every student is required to **present** at least **2** of the oral problems at the blackboard to be admitted to the exam.
- Problems marked with an asterisk ($*$) are optional and can earn you bonus points.
- If you have questions regarding the problem sets, feel free to contact your tutor at any time.

Problem 1.1: Gaussian integrals

[**Written** | 7 pt(s)]

ID: ex_gaussian_integrals_qft:qft24

Learning objective

Gaussian integrals are the essence of working with the path integral formulation of quantum mechanics. In this exercise we will recap them from the simplest $1D$ up to the N-dimensional complex case.

a) Use your favorite integration technique to solve the integral **1**pt(s)

$$\int_{-\infty}^{\infty} dx e^{-\frac{a}{2}x^2}. \tag{1}$$

What is the restriction on $a \in \mathbb{R}$ such that the integral converges?

b) Use the result of task a) to solve **1**pt(s)

$$\int_{-\infty}^{\infty} dx e^{-\frac{a}{2}x^2+bx}. \tag{2}$$

We now want to generalize this to the N-dimensional integration variable $\phi \in \mathbb{R}^N$.

c) Show that **2**pt(s)

$$\int_{\mathbb{R}^N} d^N\phi e^{-\frac{1}{2}\phi^T A\phi} = \frac{(2\pi)^{N/2}}{\sqrt{\det A}} \tag{3}$$

with $A \in \mathrm{GL}(N)$ and $A^T = A$. How does the restriction of a in task a) convert to A such that convergence is guaranteed?

d) Show that adding a source term $\phi \cdot \boldsymbol{j}$ with $\boldsymbol{j} \in \mathbb{R}^N$ leads to **1**pt(s)

$$\int_{\mathbb{R}^N} d^N\phi e^{-\frac{1}{2}\phi^T A\phi + \boldsymbol{\phi}\cdot\boldsymbol{j}} = \frac{(2\pi)^{N/2}}{\sqrt{\det A}} e^{\frac{1}{2}\boldsymbol{j}^T A^{-1}\boldsymbol{j}}. \tag{4}$$

Hint: Argue that shifting the integration variable in Eq. (3) according to $\phi = \Phi - A^{-1}\boldsymbol{j}$ does not change the result of the integral and use this equality to show Eq. (4).

Finally we generalize the Gaussian integral to the N-dimensional complex case by integrating over $\phi \in \mathbb{C}^N$. To do so we introduce the notation $\int d(z^*, z)$ which is to be understood as an integration over the whole complex plane i.e. $\int d(z^*, z) \equiv \int_{-\infty}^{\infty} d\,\mathrm{Re}\, z \int_{-\infty}^{\infty} d\,\mathrm{Im}\, z$. The $1D$ Gaussian integral can be expanded to the complex plane as

$$\begin{aligned}
\int d(z^*, z) e^{-wz^*z} &= \int_{-\infty}^{\infty} d\,\mathrm{Re}\, z \int_{-\infty}^{\infty} d\,\mathrm{Im}\, z e^{-wz^*z} \\
&= \int_{-\infty}^{\infty} da \int_{-\infty}^{\infty} db e^{-w(a-ib)(a+ib)} \\
&= \int_{-\infty}^{\infty} da \int_{-\infty}^{\infty} db e^{-w(a^2+b^2)} = \frac{\pi}{w} \qquad\qquad \text{for } \mathrm{Re}\, w > 0
\end{aligned}$$

e) Show that **2**pt(s)

$$\int d(\phi^\dagger, \phi) e^{-\phi^\dagger A\phi + \boldsymbol{j}^\dagger\boldsymbol{\phi} + \phi^\dagger\boldsymbol{j'}} = \frac{\pi^N}{\det A} e^{\boldsymbol{j}^\dagger A^{-1}\boldsymbol{j'}} \tag{5}$$

where $\boldsymbol{j}, \boldsymbol{j'} \in \mathbb{C}^N$ and $A = A^\dagger$. In N dimensions the notation is to be understood as

$$\int d(\phi^\dagger, \phi) \equiv \int_{-\infty}^{\infty} d\mathrm{Re}\phi_1 \int_{-\infty}^{\infty} d\mathrm{Im}\phi_1 \int_{-\infty}^{\infty} d\mathrm{Re}\phi_2 \int_{-\infty}^{\infty} d\mathrm{Im}\phi_2 \cdots \int_{-\infty}^{\infty} d\mathrm{Re}\phi_N \int_{-\infty}^{\infty} d\mathrm{Im}\phi_N \tag{6}$$

Hint: In the complex case $\phi^\dagger = \Phi^\dagger - \boldsymbol{j}^\dagger A^{-1}$ and $\phi = \Phi - A^{-1}\boldsymbol{j'}$ can be shifted independently (note the $\boldsymbol{j}$ and $\boldsymbol{j'}$).

Problem 1.2: Free particle propagator with path integral [**Oral** | 2 pt(s)]

ID: ex_path_integral_free_particle:qft24

Learning objective

As a first application of path integrals, we are going to consider the canonical example of a free particle. Our objective is to calculate the same propagator typically obtained as the Green's function of the Schrödinger equation, by using the knowledge acquired in the first exercise about Gaussian integrals.

Consider a free particle in one dimension with the Hamiltonian given by

$$\hat{H} = \frac{\hat{p}^2}{2m}, \tag{7}$$

with the usual momentum operator $\hat{p}$ and mass m.

Show that the free particle propagator is given by

$$\mathcal{T}_{(x,t)\to(x',t')} = i\hbar G\left(x',t';x,t\right) = \sqrt{\frac{m}{2\pi i\hbar\left(t'-t\right)}}\exp\left[\frac{im\left(x'-x\right)^2}{2\hbar\left(t'-t\right)}\right] \tag{8}$$

using the path integral approach.