Dr. Nicolai Lang
Institute for Theoretical Physics III, University of Stuttgart

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Problem 9.1: Rosenbluth formula

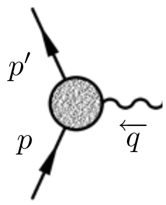
[**Oral** | 3 pt(s)]

ID: ex_rosenbluth_formula:qft23

Learning objective

In this problem, you study the scattering of an electron off a proton including the radiative corrections due to the strong interactions of the proton. You derive the scattering cross section which can be used to measure the form factors reflecting the structure of the proton.

As discussed in the lecture, the electromagnetic interaction vertex for a Dirac fermion can be written quite generally in terms of two form factors $F_1(q^2)$ and $F_2(q^2)$:



$$= \bar{u}(p') \left[\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{2m} F_2(q^2) \right] u(p)$$

where $q = p' - p$ and $\sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu]$. If the fermion is a strongly interacting particle such as the proton, the form factors reflect the structure that results from the strong interactions and so are not easy to compute from first principles. However, these form factors can be determined experimentally. In this problem, we will derive an expression for the cross section for the elastic scattering of an electron from a proton initially at rest to leading order in α but to all orders in the strong interactions.

a) Begin by first drawing the Feynman diagram of the scattering process **1**pt(s)

$$\text{proton}(p) + \text{electron}(k) \longrightarrow \text{proton}(p') + \text{electron}(k') \tag{1}$$

and write down the scattering amplitude for the process using the above rule for the vertex of the proton. The mass of the proton is m and the mass of the electron is m_e.

b) Compute the spin averaged and spin summed amplitude squared and evaluate all traces. **1**pt(s)
Hint: Use the *Gordon identity*

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p') \left[\frac{(p'+p)^\mu}{2m} + \frac{i\sigma^{\mu\nu} q_\nu}{2m} \right] u(p) \tag{2}$$

in order to simplify the evaluation of the traces. The final result can be brought into the following form:

$$\frac{1}{4} \sum |\mathcal{M}|^2 = \frac{8e^4}{q^4} \Bigg[A \Big((k' \cdot p)(k \cdot p) + (k' \cdot p)(k \cdot p') - (k \cdot k') m^2 - (p' \cdot p) m_e^2 + 2 m_e^2 m^2 \Big) + B \Big(k' \cdot (p'+p) k \cdot (p'+p) - \frac{(p'+p)^2}{2} (k' \cdot k - m_e^2) \Big) \Bigg], \tag{3}$$

where A and B have to be determined.

c) Go into the lab frame, where the proton is at rest initially. Consider an electron with energy $E \gg m_e$ (i.e. let $m_e \to 0$) and calculate the so-called *Rosenbluth formula* 1pt(s)

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2 \left[(F_1^2 - \frac{q^2}{4m^2}F_2^2)\cos^2\frac{\theta}{2} - \frac{q^2}{2m^2}(F_1+F_2)^2\sin^2\frac{\theta}{2}\right]}{2E^2\left[1+\frac{2E}{m}\sin^2\frac{\theta}{2}\right]\sin^4\frac{\theta}{2}}\,, \tag{4}$$

where θ is the lab-frame scattering angle and F_1 and F_2 are to be evaluated at the q^2 associated with elastic scattering at this angle. By measuring $\frac{d\sigma}{d\cos\theta}$ as a function of the scattering angle and different q^2, it is thus possible to extract F_1 and F_2 experimentally.

Problem 9.2: Feynman parameters

[**Written** | 3 pt(s)]

ID: ex_feynman_parameters:qft23

Learning objective

The purpose of this problem is to familiarize with the concept of Feynman parameters. It is for example used for the calculation of integrals appearing in loop diagrams and introduces additional parameters in order to bring the integrals into a form more suitable for calculation.

a) Begin with the simple case of two factors in the denominator: 1pt(s)

$$\frac{1}{AB} = \int_0^1 dx\,\frac{1}{[xA+(1-x)B]^2} = \int_0^1 dx\,dy\,\delta(x+y-1)\,\frac{1}{[xA+yB]^2}\,. \tag{5}$$

By differentiating with respect to B, prove that

$$\frac{1}{AB^n} = \int_0^1 dy\,\delta(x+y-1)\,\frac{ny^{n-1}}{[xA+yB]^{n+1}}\,. \tag{6}$$

b) Use Eq. (6) and show by induction that 1pt(s)

$$\frac{1}{A_1A_2\cdots A_n} = \int_0^1 dx_1\,dx_2\cdots dx_n\,\delta\Big(\sum x_i - 1\Big)\,\frac{(n-1)!}{[x_1A_1+x_2A_2+\cdots x_nA_n]^n}\,. \tag{7}$$

c) By repeated differentiation of (7), prove 1pt(s)

$$\frac{1}{A_1^{m_1}A_2^{m_2}\cdots A_n^{m_n}} = \int_0^1 dx_1\,dx_2\,\cdots dx_n\,\delta\Big(\sum x_i - 1\Big)\,\frac{\prod x_i^{m_i-1}}{[\sum x_iA_i]^{\sum m_i}}\,\frac{\Gamma(m_1+\cdots+m_n)}{\Gamma(m_1)\cdots\Gamma(m_n)}\,, \tag{8}$$

with $\Gamma(x)$ the gamma function and $\Gamma(n) = (n-1)!$ for positive integer n.