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Problem 6.1: Feynman diagrams for ϕ^4-theory

[**Written** | 4 (+1 bonus) pt(s)]

ID: ex_feynman_diagrams_phi4_theory:qft23

Learning objective

The purpose of this problem is to become familiar with Feynman diagrams and their corresponding perturbative expressions. To this end, we use the interacting ϕ^4-theory and focus on its four-point correlator to apply the machinery of real- and momentum-space Feynman diagrams.

We consider the ϕ^4-theory

$$H = \frac{1}{2}\int \mathrm{d}^3\boldsymbol{x}\left[\pi^2(\boldsymbol{x}) + (\nabla\phi(\boldsymbol{x}))^2 + m^2\phi^2(\boldsymbol{x}) + 2\,\frac{\lambda}{4!}\phi^4(\boldsymbol{x})\right] \tag{1}$$

with interacting fields $\phi(x) = e^{iHt}\phi(\boldsymbol{x})e^{-iHt}$ and vacuum $|\Omega\rangle$.

a) Draw all *relevant* Feynman diagrams (i.e., without vacuum bubbles) for the perturbative expansion of the four-point function **1**pt(s)

$$\langle\Omega|\,\mathcal{T}\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\,|\Omega\rangle \tag{2}$$

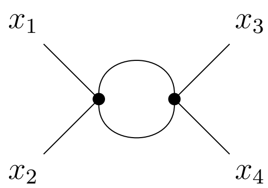
up to second order (λ^2).

Draw two relevant diagrams of third order (λ^3): one connected and one disconnected.

Hint: Ignore symmetry factors and permutations of external points. Use that four-point diagrams are either fully connected or decompose into products of disjoint two-point diagrams. Up to permutations, there are 3 connected diagrams and 6 additional disconnected diagrams up to second order.

*b) Draw all diagrams of third order. How many are connected and disconnected, respectively (again up to permutations)? **+1**pt(s)

c) Using the *real-space Feynman rules*, write down the term described by the Feynman diagram **1**pt(s)



d) Label the Feynman diagram above with directed momenta and write down the corresponding expression as prescribed by the *momentum-space Feynman rules.* **1**pt(s)

e) Use the Fourier expansion of the Feynman propagator **1**pt(s)

$$D_F(x-y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\,\frac{i\,e^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon} \tag{3}$$

to show that the expressions of c) and d) are equivalent.

Problem 6.2: Feynman rules for the interacting complex Klein-Gordon field [**Oral** | 4 pt(s)]

ID: ex_feynman_rules_interacting_complex_klein_gordon_field:qft23

Learning objective

Here you derive the Feynman rules for the complex Klein-Gordon field with an arbitrary interaction potential. Generically, this interaction violates causality and the resulting theory is no longer a relativistic quantum field theory. However, in condensed matter physics such theories can be used to describe the low-energy physics of interacting models that are otherwise hard to tackle analytically. This demonstrates that diagrammatic methods for perturbation theory are not restricted to relativistic high-energy physics.

Recall the (free) complex Klein-Gordon field (Problem 2.2) with Hamiltonian

$$H_0 = \int \mathrm{d}^3\boldsymbol{x}\, \left(\pi^\dagger\pi + \nabla\phi^\dagger\nabla\phi + m^2\phi^\dagger\phi\right) \tag{4}$$

and fields that satisfy the canonical commutation relations $[\phi(\boldsymbol{x}),\pi(\boldsymbol{y})] = i\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})$.

Let $V\,:\,\mathbb{R}^3\to\mathbb{R}$ be a symmetric $[V(\boldsymbol{r}) = V(-\boldsymbol{r})]$ but otherwise arbitrary (well-behaved) potential. Here we consider the interacting theory

$$H = H_0 + \frac{\lambda}{2}\int \mathrm{d}^3\boldsymbol{x}\int \mathrm{d}^3\boldsymbol{y}\, V(\boldsymbol{x}-\boldsymbol{y})\,\phi^\dagger(\boldsymbol{x})\phi^\dagger(\boldsymbol{y})\phi(\boldsymbol{x})\phi(\boldsymbol{y}) \tag{5}$$

with small parameter λ.

At an arbitrary time t_0, we can expand the interacting field $\phi(t_0,\boldsymbol{x})$ into modes,

$$\phi(t_0,\boldsymbol{x}) = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}\left(a_{\boldsymbol{p}}e^{i\boldsymbol{p}\boldsymbol{x}} + b^\dagger_{\boldsymbol{p}}e^{-i\boldsymbol{p}\boldsymbol{x}}\right)\,, \tag{6}$$

with the mode algebra

$$\left[a_{\boldsymbol{p}},a^\dagger_{\boldsymbol{q}}\right] = (2\pi)^3\,\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q}) \quad \text{and} \quad \left[b_{\boldsymbol{p}},b^\dagger_{\boldsymbol{q}}\right] = (2\pi)^3\,\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q}) \tag{7}$$

(all other commutators vanish). In the interaction picture, we then have

$$\phi_I(x) = e^{iH_0(t-t_0)}\phi(t_0,\boldsymbol{x})e^{-iH_0(t-t_0)} = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}\left(a_{\boldsymbol{p}}e^{-ipx} + b^\dagger_{\boldsymbol{p}}e^{ipx}\right) \tag{8}$$

with $x^0 = t - t_0$. Note that this is just the time evolution of the free theory H_0 that you derived in Problem 2.2 b).

a) Let the contraction be defined as difference between time ordering and normal ordering: 1 pt(s)

$$\overbracket{AB} \equiv \mathcal{T}\{AB\} - {:}AB{:} \tag{9}$$

where $A,B\in\{\phi_I,\phi_I^\dagger\}$.

Use the decomposition $\phi_I = \phi_a^+ + \phi_b^-$ and $\phi_I^\dagger = \phi_a^- + \phi_b^+$ into positive- and negative-frequency parts (and your knowledge from the real Klein-Gordon field) to show that

$$\overbracket{\phi_I(x)\phi_I}(y) = \overbracket{\phi_I^\dagger(x)\phi_I^\dagger}(y) = 0 \tag{10a}$$

$$\overbracket{\phi_I^\dagger(x)\phi_I}(y) = \overbracket{\phi_I(x)\phi_I^\dagger}(y) = D_F(x-y) = \int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{i\,e^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon}\,. \tag{10b}$$

b) Prove Wick's theorem for the free complex scalar field. That is, show that **1**pt(s)

$$\mathcal{T}\{ABC\dots\} = {:}ABC\dots{:} + {:}\{\text{all contractions between pairs of } \phi \text{ and } \phi^\dagger\}{:} \tag{11}$$

for $A, B, C, \dots \in \{\phi_I, \phi_I^\dagger\}$.

Hint: Use induction (as in Peskin & Schroeder) with the decomposition of ϕ and $\phi^\dagger$ from above.

c) As shown in the lecture (or in Problem 5.1), time-ordered correlation functions can be rewritten in terms of interaction picture fields via **1**pt(s)

$$\langle\Omega|\,\mathcal{T}\{ABC\dots\}\,|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \frac{\langle 0|\,\mathcal{T}\{A_I B_I C_I \dots \exp\left(-i\int_{-T}^{T} dt\, H_I(t)\right)\}\,|0\rangle}{\langle 0|\,\mathcal{T}\exp\left(-i\int_{-T}^{T} dt\, H_I(t)\right)|0\rangle} \tag{12}$$

for $A, B, C, \dots \in \{\phi, \phi^\dagger\}$. Here $|\Omega\rangle$ is the interacting vacuum and the interaction picture Hamiltonian is given by

$$H_I(t) = \frac{\lambda}{2}\int \mathrm{d}^3\boldsymbol{x} \int \mathrm{d}^3\boldsymbol{y}\, V(\boldsymbol{x}-\boldsymbol{y})\,\phi_I^\dagger(x)\phi_I^\dagger(y)\phi_I(x)\phi_I(y)\,. \tag{13}$$

Use this prescription in combination with Wick's theorem to evaluate the two-point correlator

$$\langle\Omega|\,\mathcal{T}\phi(x)\phi^\dagger(y)\,|\Omega\rangle \tag{14}$$

up to first order in λ.

Compare your result to the ϕ^4-theory.

d) Use the dictionary **1**pt(s)

$$y \longrightarrow x \;=\; \contraction{}{\phi}{_I(x)}{\phi}\phi_I(x)\phi_I^\dagger(y) = D_F(x-y) \tag{15a}$$

$$u \;\text{- - - - - - -}\; w \;=\; V(\boldsymbol{u}-\boldsymbol{w})\,\delta(u^0-w^0) \tag{15b}$$

to recast the summands found in c) as Feynman diagrams.

Generalize your result to the Feynman rules of the interacting theory of a complex scalar field with interaction potential V.