Problem 6.1: Feynman diagrams for $\phi^4$-theory

Learning objective
The purpose of this problem is to become familiar with Feynman diagrams and their corresponding perturbative expressions. To this end, we use the interacting $\phi^4$-theory and focus on its four-point correlator to apply the machinery of real- and momentum-space Feynman diagrams.

We consider the $\phi^4$-theory

$$H = \frac{1}{2} \int d^3x \left[ \pi^2(x) + (\nabla \phi(x))^2 + m^2 \phi^2(x) + 2 \frac{\lambda}{4!} \phi^4(x) \right]$$

with interacting fields $\phi(x) = e^{iHt} \phi(x) e^{-iHt}$ and vacuum $|\Omega\rangle$.

a) Draw all relevant Feynman diagrams (i.e., without vacuum bubbles) for the perturbative expansion of the four-point function

$$\langle\Omega| T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) |\Omega\rangle$$

up to second order ($\lambda^2$).

Draw two relevant diagrams of third order ($\lambda^3$): one connected and one disconnected.

Hint: Ignore symmetry factors and permutations of external points. Use that four-point diagrams are either fully connected or decompose into products of disjoint two-point diagrams. Up to permutations, there are 3 connected diagrams and 6 additional disconnected diagrams up to second order.

* b) Draw all diagrams of third order. How many are connected and disconnected, respectively (again up to permutations)?

c) Using the real-space Feynman rules, write down the term described by the Feynman diagram

![Feynman Diagram](image)

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d) Label the Feynman diagram above with directed momenta and write down the corresponding expression as prescribed by the momentum-space Feynman rules.

e) Use the Fourier expansion of the Feynman propagator

$$D_F(x - y) = \int \frac{d^4p}{(2\pi)^4} \frac{i e^{-ip(x-y)}}{p^2 - m^2 + i\epsilon}$$

(3)

to show that the expressions of c) and d) are equivalent.
Problem 6.2: Feynman rules for the interacting complex Klein-Gordon field  

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Problem Set 6

Problem 6.2: Feynman rules for the interacting complex Klein-Gordon field  

[Oral | 4 pt(s)]

Learning objective

Here you derive the Feynman rules for the complex Klein-Gordon field with an arbitrary interaction potential. Generically, this interaction violates causality and the resulting theory is no longer a relativistic quantum field theory. However, in condensed matter physics such theories can be used to describe the low-energy physics of interacting models that are otherwise hard to tackle analytically. This demonstrates that diagrammatic methods for perturbation theory are not restricted to relativistic high-energy physics.

Recall the (free) complex Klein-Gordon field (Problem 2.2) with Hamiltonian

\[ H_0 = \int d^3x \left( \pi^+ \pi + \nabla \phi^\dagger \nabla \phi + m^2 \phi^\dagger \phi \right) \tag{4} \]

and fields that satisfy the canonical commutation relations \([\phi(x), \pi(y)] = i\delta^{(3)}(x - y)\).

Let \( V : \mathbb{R}^3 \rightarrow \mathbb{R} \) be a symmetric \([V(r) = V(-r)]\) but otherwise arbitrary (well-behaved) potential. Here we consider the interacting theory

\[ H = H_0 + \frac{\lambda}{2} \int d^3x \int d^3y \, V(x - y) \phi^\dagger(x) \phi^\dagger(y) \phi(x) \phi(y) \tag{5} \]

with small parameter \(\lambda\).

At an arbitrary time \(t_0\), we can expand the interacting field \(\phi(t_0, x)\) into modes,

\[ \phi(t_0, x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left( a_p e^{ipx} + b_p^\dagger e^{-ipx} \right) , \tag{6} \]

with the mode algebra

\[ [a_p, a_q^\dagger] = (2\pi)^3 \delta^{(3)}(p - q) \quad \text{and} \quad [b_p, b_q^\dagger] = (2\pi)^3 \delta^{(3)}(p - q) \tag{7} \]

(all other commutators vanish). In the interaction picture, we then have

\[ \phi_I(x) = e^{iH_0(t-t_0)}\phi(t_0, x)e^{-iH_0(t-t_0)} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left( a_p e^{ipx} + b_p^\dagger e^{-ipx} \right) \tag{8} \]

with \(x^0 = t - t_0\). Note that this is just the time evolution of the free theory \(H_0\) that you derived in Problem 2.2 b).

a) Let the contraction be defined as difference between time ordering and normal ordering:

\[ \overline{AB} \equiv T\{AB\} : AB : \tag{9} \]

where \(A, B \in \{\phi_I, \phi_I^\dagger\}\).

Use the decomposition \(\phi_I = \phi_I^+ + \phi_I^-\) and \(\phi_I^\dagger = \phi_I^+ + \phi_I^-\) into positive- and negative-frequency parts (and your knowledge from the real Klein-Gordon field) to show that

\[ \overline{\phi_I(x)\phi_I(y)} = \overline{\phi_I^+ (x)\phi_I^+ (y)} = 0 \tag{10a} \]

\[ \overline{\phi_I^\dagger (x)\phi_I (y)} = \overline{\phi_I (x)\phi_I^\dagger (y)} = D_F(x - y) = \int \frac{d^4p}{(2\pi)^4} \frac{ie^{-ip(x-y)}}{p^2 - m^2 + i\epsilon} . \tag{10b} \]
b) Prove Wick’s theorem for the free complex scalar field. That is, show that

\[ T \{ ABC \ldots \} = \{ all \text{ contractions between pairs of } \phi \text{ and } \phi^\dagger \} : \]

for \( A, B, C, \cdots \in \{ \phi_I, \phi_I^\dagger \} \).

**Hint:** Use induction (as in Peskin & Schroeder) with the decomposition of \( \phi \) and \( \phi^\dagger \) from above.

c) As shown in the lecture (or in Problem 5.1), time-ordered correlation functions can be rewritten in terms of interaction picture fields via

\[
\langle \Omega | T \{ ABC \ldots \} | \Omega \rangle = \lim_{T \to \infty (1 - i\epsilon)} \frac{\langle 0 | T \{ A_I B_I C_I \ldots \ exp \left( -i \int_{-T}^{T} dt H_I(t) \right) \} | 0 \rangle}{\langle 0 | T \exp \left( -i \int_{-T}^{T} dt H_I(t) \right) | 0 \rangle} \]

for \( A, B, C, \cdots \in \{ \phi, \phi^\dagger \} \). Here \( | \Omega \rangle \) is the interacting vacuum and the interaction picture Hamiltonian is given by

\[
H_I(t) = \frac{\lambda}{2} \int d^3 x \int d^3 y \, V(x - y) \, \phi^\dagger_I(x) \phi^\dagger_I(y) \phi_I(x) \phi_I(y). \]

Use this prescription in combination with Wick’s theorem to evaluate the two-point correlator

\[
\langle \Omega | T \phi(x) \phi^\dagger(y) | \Omega \rangle \]

up to first order in \( \lambda \).

Compare your result to the \( \phi^4 \)-theory.

d) Use the dictionary

\[
y \rightarrow x = \phi_I(x) \phi_I^\dagger(y) = D_F(x - y) \]  \hspace{1cm} \text{(15a)}
\[
u \rightarrow w = V(u - w) \delta(u^0 - w^0) \]  \hspace{1cm} \text{(15b)}

and recast the summands found in c) as Feynman diagrams.

Generalize your result to the Feynman rules of the interacting theory of a complex scalar field with interaction potential \( V \).