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Problem 5.1: Perturbation expansion of correlation functions

[**Written** | 5 pt(s)]

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Learning objective

The purpose of this problem is to study the time evolution operator which relates the two-point correlation function of an interacting quantum field theory to a correlation function of the corresponding non-interacting theory. This correlation function then can be evaluated by means of perturbation theory leading to Feynman diagrams.

We start by considering a generic Hamiltonian

$$H = H_0 + H_{\text{int}}(\lambda)\,, \tag{1}$$

where H_0 is the Hamiltonian of some exactly solvable, non-interacting quantum field theory and $H_{\text{int}}(\lambda)$ is some perturbation depending on the (small) parameter λ. For the moment, you might think of them as H_0 being the Hamiltonian of the free, real Klein-Gordon field and H_{int} as the Hamiltonian of the ϕ^4-theory. In the end, we are interested in the two-point correlation function

$$\langle\Omega|\,\mathcal{T}\phi(x)\phi(y)\,|\Omega\rangle\;, \tag{2}$$

where $|\Omega\rangle$ is the vacuum of the full Hamiltonian H and $\mathcal{T}$ denotes the time-ordering operator. The influence of the interaction part of the Hamiltonian is encoded in the field ϕ as well as in the vacuum $|\Omega\rangle$ and the goal of this problem is to study its relation to the operators of the free field as well as the vacuum of the non-interacting theory, $|0\rangle$ with $H_0\,|0\rangle = 0$.

First, recall that in the *Heisenberg picture* the field operator for some time $t \neq t_0$ takes the form

$$\phi(t,\boldsymbol{x}) = e^{iH(t-t_0)}\phi(t_0,\boldsymbol{x})e^{-iH(t-t_0)} \tag{3}$$

with $\phi(t_0,\boldsymbol{x})$ being the field operator in the Schrödinger picture.

In the *interaction picture*, the field takes the form

$$\phi_I(t,\boldsymbol{x}) = e^{iH_0(t-t_0)}\phi(t_0,\boldsymbol{x})e^{-iH_0(t-t_0)}\,. \tag{4}$$

Defining the *time evolution operator*

$$U(t,t_0) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)}\,, \tag{5}$$

we can relate the field operator in the Heisenberg picture to the field operator in the interaction picture by

$$\phi(t,\boldsymbol{x}) = U^\dagger(t,t_0)\phi_I(t,\boldsymbol{x})U(t,t_0)\,. \tag{6}$$

a) Show that the time evolution operator (5) satisfies the differential equation 1pt(s)

$$i\partial_t U(t,t_0) = H_I(t)U(t,t_0)\,, \tag{7}$$

where $H_I(t)$ refers to H_{int} in the interaction picture.

b) Show that the solution of this differential equation is given by 1pt(s)

$$\begin{aligned} U(t,t_0) = 1 &+ (-i)\int_{t_0}^{t} dt'\, H_I(t') + (-i)^2 \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''\, H_I(t')H_I(t'') \\ &+ (-i)^3 \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \int_{t_0}^{t''} dt'''\, H_I(t')H_I(t'')H_I(t''') + \cdots \end{aligned} \tag{8}$$

and that this can be written in the form

$$U(t,t_0) = \mathcal{T} \exp\left(-i\int_{t_0}^{t} dt'\, H_I(t')\right)\,. \tag{9}$$

c) Show that the time evolution operator $U(t,t')$ satisfies the following properties: 1pt(s)

$$U^{-1}(t,t') = U^\dagger(t,t') \tag{10a}$$

$$U(t_1,t_2)U(t_2,t_3) = U(t_1,t_3)\,. \tag{10b}$$

d) Now study the ground state $|\Omega\rangle$ of the full, interacting theory. Isolate the ground state $|\Omega\rangle$ by evolving the state $|0\rangle$ with the full Hamiltonian H to some time T and show that 1pt(s)

$$|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \left(e^{-iE_0(t_0+T)}\langle\Omega|0\rangle\right)^{-1} U(t_0,-T)\,|0\rangle\,, \tag{11}$$

where $E_0 = \langle\Omega|\,H\,|\Omega\rangle$, t_0 is a small constant, and we assume that $\langle\Omega|0\rangle \neq 0$.

Hint: Expand $|0\rangle$ into eigenstates $|n\rangle$ of the full Hamiltonian and project out the interacting ground state by pushing the time into a slightly imaginary direction, $T\to\infty(1-i\varepsilon)$.

e) Consider first the case $x^0 > y^0$ and show that the two-point correlation function (2) can be written in terms of the non-interacting vacuum as 1pt(s)

$$\langle\Omega|\,\phi(x)\phi(y)\,|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \frac{\langle 0|\,U(T,x^0)\phi_I(x)U(x^0,y^0)\phi_I(y)U(y^0,-T)\,|0\rangle}{\langle 0|\,U(T,-T)\,|0\rangle}\,. \tag{12}$$

Show that this generalizes to

$$\langle\Omega|\,\mathcal{T}\phi(x)\phi(y)\,|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \frac{\langle 0|\,\mathcal{T}\phi_I(x)\phi_I(y)\exp\left(-i\int_{-T}^{T} dt\, H_I(t)\right)|0\rangle}{\langle 0|\,\mathcal{T}\exp\left(-i\int_{-T}^{T} dt\, H_I(t)\right)|0\rangle}\,. \tag{13}$$

Problem 5.2: Wick's theorem

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ID: ex_wicks_theorem:qft23

Learning objective

In this problem, you will prove Wick's theorem by constructing a generating functional. Wick's theorem allows to reorder a time-ordered product of operators, as those appearing in the two-point correlation function studied in Problem 5.1, into a normal form which is more convenient for the actual computation in Fock space.

In the following, we only deal with free fields and assume that all fields are in the interaction picture and therefore drop the subscript I. Further, we restrict ourselves to the real Klein-Gordon field which is bosonic, noting that Wick's theorem can be derived similarly for fermionic fields. In order to derive the generating functional, we use the Hamiltonian

$$H_I(t) = \int d^3x\, j(t, \boldsymbol{x})\phi(t, \boldsymbol{x}) \tag{14}$$

where the real scalar field ϕ is coupled to the source term $j(x)$.

a) First, consider two operators A and B with $[A, [A, B]] = [B, [A, B]] = 0$. Prove the *Baker-Campbell-Hausdorff formula* 1pt(s)

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]}\,. \tag{15}$$

Hint: Start with $f(t) = e^{tA}e^{tB}$ and derive a differential equation for $f(t)$ which you can solve. For $t = 1$, you will recover (15).

b) Now, look at the operator 1pt(s)

$$S = U(\infty, -\infty) = \mathcal{T} \exp\left(-i \int dt\, H_I(t)\right) = \lim_{\substack{t_i\to-\infty \\ t_f\to\infty}} \mathcal{T} \exp\left(-i \int_{t_i}^{t_f} dt\, H_I(t)\right)\,. \tag{16}$$

Show that the commutator $[H_I(t), H_I(t')]$ is a complex number and show that the operator S can be rewritten as

$$S = \exp\left(-i \int dt\, H_I(t)\right) \exp\left(-\frac{1}{2} \int dt\, dt'\, \theta(t - t')[H_I(t), H_I(t')]\right)\,, \tag{17}$$

where $\theta(t)$ is the Heaviside function.

Hint: Discretize the interval $[t_i, t_f]$ and rewrite the time-ordered exponential as a product of exponentials.

c) Expand the scalar field ϕ into its positive- and negative-frequency part, $\phi(x) = \phi^{(+)}(x)+\phi^{(-)}(x)$, and show that 1pt(s)

$$\begin{aligned} S = {}&:\!\exp\left(-i \int d^4x\, \phi(x)j(x)\right)\!: \\ &\times \exp\left(\frac{1}{2} \int d^4x\, d^4y\, j(x) \left([\phi^{(-)}(x), \phi^{(+)}(y)] - \theta(x^0 - y^0)[\phi(x), \phi(y)]\right) j(y)\right)\,. \end{aligned} \tag{18}$$

Hint: $:\!e^{a+a^\dagger}\!: = e^{a^\dagger}e^a$

d) Using the fact that the commutator for free fields is just a complex number, show that (18) can be further simplified to **1**pt(s)

$$S = :\exp\left(-i\int d^4x\, j(x)\phi(x)\right): \exp\left(-\frac{1}{2}\int d^4x\, d^4y\, j(x)j(y)\, \langle 0|\, \mathcal{T}\phi(x)\phi(y)\, |0\rangle\right)\,. \tag{19}$$

e) Finally, use (19) to prove *Wick's theorem* **1**pt(s)

$$\mathcal{T}\phi(x_1)\phi(x_2)\cdots\phi(x_m) = :\phi(x_1)\phi(x_2)\cdots\phi(x_m) + \text{all possible contractions}:\,. \tag{20}$$

The *contraction* of two fields is defined as

$$\overbrace{\phi(x)\phi}(y) = \begin{cases} [\phi^{(+)}(x), \phi^{(-)}(y)] & \text{for } x^0 > y^0\,, \\ [\phi^{(+)}(y), \phi^{(-)}(x)] & \text{for } y^0 > x^0\,, \end{cases} \tag{21}$$

and it is

$$\overbrace{\phi(x)\phi}(y) = D_F(x-y) \tag{22}$$

with the Feynman propagator $D_F(x-y)$.