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Problem 3.1: Fock states and coherent states

[**Oral** | 4 (+1 bonus) pt(s)]

ID: ex_fock_states_coherent_states:qft23

Learning objective

In this problem, we construct the Hilbert space of a (bosonic) quantum field theory on the basis of linear superpositions of non-normalizable single-particle states and discuss how the concept of coherent states translates to this setting.

The single-particle states $|\boldsymbol{p}\rangle = \sqrt{2E_{\boldsymbol{p}}}a^\dagger_{\boldsymbol{p}}|0\rangle$ are not well suited to build up the Hilbert space as they are not normalizable due their diverging commutation relations, $[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}] = (2\pi)^3\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})$. However, it is possible to build a wave packet by linear superposition of momentum eigenstates,

$$a^\dagger(f)|0\rangle = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}}}} f(\boldsymbol{p}) a^\dagger_{\boldsymbol{p}}|0\rangle\ . \tag{1}$$

a) Calculate the commutator $[a(f), a^\dagger(f)]$ and derive a condition for $f(\boldsymbol{p})$ such that the states are normalizable. **1**pt(s)

b) Consider now the generalization to n particles. We define the unnormalized *Fock state* as **1**pt(s)

$$|n\rangle = \int \frac{d^3p_1}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}_1}}} \cdots \int \frac{d^3p_n}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}_n}}} F(\boldsymbol{p}_1,\dots,\boldsymbol{p}_n) a^\dagger_{\boldsymbol{p}_1}\cdots a^\dagger_{\boldsymbol{p}_n}|0\rangle\ , \tag{2}$$

where F is symmetric under the exchange of two of its arguments. Calculate the norm of (2). Show that (2) is an eigenstate of the number operator

$$N = \int \frac{d^3p}{(2\pi)^3} a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}} \tag{3}$$

and calculate its eigenvalue.

c) Calculate the expectation value of the (normal ordered) Hamiltonian **1**pt(s)

$$H = \int \frac{d^3p}{(2\pi)^3} E_{\boldsymbol{p}} a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}} \tag{4}$$

of the real Klein-Gordon field with respect to the state (2).

d) Consider now a coherent superposition of n-particle states, that is a *coherent state*, **1**pt(s)

$$|\alpha\rangle = \exp\left(\int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}}}} \alpha(\boldsymbol{p}) a^\dagger_{\boldsymbol{p}}\right)|0\rangle\ . \tag{5}$$

Calculate its norm as well as the overlap of two (normalized) coherent states, $\langle\alpha|\beta\rangle$. Interpret the result.

*e) Show that (5) is an eigenstate of the annihilation operator $a_{\boldsymbol{p}}$ and calculate its eigenvalue. Show that the coherent state remains a coherent state under time evolution with the Hamiltonian (4), that is, show that $e^{-iHt}|\alpha\rangle = |\beta\rangle$, where $|\beta\rangle$ has to be determined. **+1** pt(s)

Problem 3.2: Free-particle solutions of the Dirac equation [**Written** | 4 pt(s)]

ID: ex_free_particle_solutions_dirac_equation:qft23

Learning objective

In this problem, you will fill in the missing calculations of the discussion of the free-particle solutions of the Dirac equation in the lecture. Additionally, you will calculate two important completeness relations which will prove useful in the evaluation of Feynman diagrams.

In the lecture, you discussed that the general solution of the Dirac equation can be written as a superposition of plane waves whose positive-frequency solutions are given by

$$\psi(x) = u(p)e^{-ipx}\,, \qquad p^2 = m^2\,, \qquad p^0 > 0\,. \tag{6}$$

The two independent solutions for $u(p)$ read

$$u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix}\,, \qquad s = 1,2 \tag{7}$$

with the normalization $(\xi^s)^\dagger\xi^r = \delta^{rs}$ of the two-component spinor ξ^s. Here, $\sigma^\mu = (1, \boldsymbol{\sigma})$ and $\bar{\sigma}^\mu = (1, -\boldsymbol{\sigma})$ with the Pauli matrices σ_i.

a) Prove the identity $(p\cdot\sigma)(p\cdot\bar{\sigma}) = p^2 = m^2$. Also show that $(\gamma^\mu p_\mu + m)(\gamma^\mu p_\mu - m) = 0$. **1** pt(s)

b) Show that $u^{r\dagger}(p)u^s(p)$ is not Lorentz invariant. Show that instead $u(p)$ can be normalized in a Lorentz invariant way according to **1** pt(s)

$$\bar{u}^r(p)u^s(p) = 2m\delta^{rs} \tag{8}$$

with $\bar{u} = u^\dagger\gamma^0$.

c) Similarly to above, the negative-frequency solutions **1** pt(s)

$$\psi(x) = v(p)e^{ipx}\,, \qquad p^2 = m^2\,, \qquad p^0 > 0 \tag{9}$$

can be obtained with two linearly independent solutions

$$v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\,\eta^s \end{pmatrix}\,, \qquad s = 1,2\,, \tag{10}$$

where η^s is another basis of two-component spinors. These solutions are normalized according to $\bar{v}^r(p)v^s(p) = -2m\delta^{rs}$.

Show that $\bar{u}^r(p)v^s(p) = \bar{v}^r(p)u^s(p) = 0$ and $u^{r\dagger}(p)v^s(p) \neq 0$ as well as $v^{r\dagger}(p)u^s(p) \neq 0$. However, show that reversing the sign of the 3-momentum in one factor of each spinor product leads to $u^{r\dagger}(\boldsymbol{p})v^s(-\boldsymbol{p}) = v^{r\dagger}(\boldsymbol{p})u^s(-\boldsymbol{p}) = 0$.

d) Finally, we consider the sum over the polarization states of a fermion which will be important when evaluating Feynman diagrams. Calculate the completeness relations **1**pt(s)

$$\sum_s u^s(p)\bar{u}^s(p) \quad = \not{p} + m\,, \tag{11}$$

$$\sum_s v^s(p)\bar{v}^s(p) \quad = \not{p} - m \tag{12}$$

with the Feynman slash notation $\not{p} \equiv \gamma^\mu p_\mu$.

Problem 3.3: Lorentz group

[**Oral** | 5 pt(s)]

ID: ex_lorentz_group:qft23

Learning objective

In this exercise we want to familiarize ourselves with the Lorentz group and its continuous and discrete generators.

The Lorentz group $O(1,3)$ is defined as group of 4×4 matrices Λ that keep the Minkowski metric $g = \mathrm{diag}(1,-1,-1,-1)$ invariant, i.e.

$$O(1,3) = \{\Lambda \in \mathbb{R}^{4\times4}\,|\,\Lambda^\mu{}_\alpha g_{\mu\nu}\Lambda^\nu{}_\beta = g_{\alpha\beta} \iff \Lambda^T g\Lambda = g\}\,. \tag{13}$$

a) Show that $O(1,3)$ is a group. Specifically show the following properties for any $\Lambda_1, \Lambda_2 \in O(1,3)$: **1**pt(s)
   i. $\Lambda_1\Lambda_2 \in O(1,3)$
   ii. Λ_1^{-1} exists and is in $O(1,3)$.

Since the Lorentz group is a Lie group, we can study its Lie algebra. Therefore, we look at an infinitesimal Lorentz transformation

$$\Lambda_{\boldsymbol{\xi}} = \exp(-i\xi_i X_i) \overset{\xi_i \ll 1}{\approx} 1 - i\xi_i X_i\,, \tag{14}$$

where the $X_i \in \mathbb{R}^{4\times4}$ define the Lie algebra and are the generators of the Lorentz group and ξ_i are the corresponding coefficients.

b) How does the condition of the Lorentz group $\Lambda^T g\Lambda = g$ translate to the generators X_i? Show that we can write the generators X_i as **1**pt(s)

$$(X_i)^\mu{}_\nu = \begin{pmatrix} 0 & a & b & c \\ a & 0 & d & e \\ b & -d & 0 & f \\ c & -e & -f & 0 \end{pmatrix}\,, \tag{15}$$

where there are only 6 degrees of freedom left.

The basis set of the Lorentz Lie algebra is often chosen as

$$(\mathcal{J}^{\mu\nu})_{\rho\sigma} = i\left(\delta^\mu{}_\rho\delta^\nu{}_\sigma - \delta^\mu{}_\sigma\delta^\nu{}_\rho\right)\,, \tag{16}$$

with

$$\Lambda_{\omega} = \exp(-\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}), \tag{17}$$

where both, $\omega_{\mu\nu}$ and $\mathcal{J}^{\mu\nu}$ are antisymmetric.

Convince yourself that this basis corresponds to the matrices (15) by choosing one of the parameters as one, and all the others zero. Therefore, the antisymmetric coefficient tensor $\omega_{\mu\nu}$ still hosts 6 degrees of freedom.

c) Show that the commutator is: **1**pt(s)

$$([\mathcal{J}^{\mu\nu}, \mathcal{J}^{\alpha\beta}])^{\rho}{}_{\sigma} = i\left(g^{\nu\alpha}\mathcal{J}^{\mu\beta} - g^{\nu\beta}\mathcal{J}^{\mu\alpha} - g^{\mu\alpha}\mathcal{J}^{\nu\beta} + g^{\mu\beta}\mathcal{J}^{\nu\alpha}\right)^{\rho}{}_{\sigma}\,. \tag{18}$$

This defines the Lorentz algebra, and it has to be the same for all representations of the Lorentz group.

d) Calculate the determinant of Λ for the definition in (13) and in (14). **1**pt(s)

e) Convince yourself, that time reversal T and space inversion P are part of the Lorentz group (13), but they cannot be continuously connected to the identity **1**pt(s)

$$T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad , \quad P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{19}$$

Finally, also prove that T and P cannot be continuously connected.

Interpret your result by sketching the structure of the Lorentz group.