

Quantum Field Theory Exercise: Lorentz covariance

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*Text in black was given as input, text in blue was produced by **ChatGPT**. I simply used the \LaTeX -code of the original exercise without any modifications as prompts. I copied the replies by **ChatGPT** without any modifications. It's \LaTeX -code compiled flawlessly (I had a few compile issues, but these were all my fault); it even used labels and references to refer to its own equations.*

Motivation This exercise serves as a **brief** revision of tensor calculus and the covariant formulation of classical electromagnetism.

Context In the following, we will work in units where $c = 1$. Further, we will make use of *Einstein notation* where summation over indices appearing twice is assumed.

We first introduce the four-vector

$$x^\mu = (t, \mathbf{r}), \quad \mu = 0, 1, 2, 3, \quad (1)$$

which we will call a **contravariant** vector or tensor of first order. The vector x_μ is called **covariant** vector. In special relativity, the metric tensor is given by

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (2)$$

and the relationship between co- and contravariant vectors is given by

$$x_\mu = g_{\mu\nu}x^\nu. \quad (3)$$

A **Lorentz vector** is an object that under a Lorentz transformation $\Lambda^\mu{}_\nu$ transforms like

$$\tilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu. \quad (4)$$

In tensors of higher order, each index transforms as a Lorentz vector, e.g.

$$\tilde{A}^{\alpha\beta\gamma}{}_{\delta\varepsilon} = \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \Lambda^\gamma{}_\xi \Lambda^\delta{}_\rho \Lambda^\varepsilon{}_\sigma A^{\mu\nu\xi}{}_{\rho\sigma}. \quad (5)$$

A **Lorentz scalar** is a quantity that is invariant under Lorentz transformations.

a) Show that $x^\mu x_\mu$ is a Lorentz scalar, i.e. show that $x^\mu x_\mu = \tilde{x}^\sigma \tilde{x}_\sigma$.

To show that $x^\mu x_\mu$ is a Lorentz scalar, we need to prove that it remains invariant under Lorentz transformations. In other words, we need to show that $x^\mu x_\mu = \tilde{x}^\sigma \tilde{x}_\sigma$.

First, recall that under a Lorentz transformation, the contravariant coordinates transform as

$$\tilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu. \quad (6)$$

Now, let's find the covariant coordinates in the transformed frame. Using the relationship between co- and contravariant vectors given by

$$x_\mu = g_{\mu\nu}x^\nu, \quad (7)$$

we can write

$$\tilde{x}_\sigma = g_{\sigma\rho}\tilde{x}^\rho. \quad (8)$$

Substitute Eq. (6) into Eq. (8) to obtain

$$\tilde{x}_\sigma = g_{\sigma\rho}\Lambda^\rho{}_\nu x^\nu. \quad (9)$$

Now, let's compute the quantity $\tilde{x}^\sigma \tilde{x}_\sigma$. Using Eqs. (6) and (9), we get

$$\tilde{x}^\sigma \tilde{x}_\sigma = (\Lambda^\sigma{}_\mu x^\mu)(g_{\sigma\rho} \Lambda^\rho{}_\nu x^\nu) \quad (10)$$

$$= \Lambda^\sigma{}_\mu x^\mu \Lambda^\rho{}_\nu g_{\sigma\rho} x^\nu \quad (11)$$

$$= x^\mu (\Lambda^\sigma{}_\mu g_{\sigma\rho} \Lambda^\rho{}_\nu) x^\nu. \quad (12)$$

Using the property of Lorentz transformations, we have

$$\Lambda^\sigma{}_\mu g_{\sigma\rho} \Lambda^\rho{}_\nu = g_{\mu\nu}. \quad (13)$$

Therefore,

$$\tilde{x}^\sigma \tilde{x}_\sigma = x^\mu g_{\mu\nu} x^\nu \quad (14)$$

$$= x^\mu x_\mu. \quad (15)$$

This result shows that $x^\mu x_\mu$ is a Lorentz scalar, as it remains invariant under Lorentz transformations.

b) Another important object is the four-gradient

$$\frac{\partial}{\partial x^\mu} = \partial_\mu = (\partial_t, \nabla). \quad (16)$$

Compute the d'Alembert operator $\partial^\mu \partial_\mu$. Is this quantity a Lorentz scalar?

The d'Alembert operator, also known as the d'Alembertian, is defined as the contraction of the four-gradient with itself:

$$\partial^\mu \partial_\mu = g^{\mu\nu} \partial_\mu \partial_\nu. \quad (17)$$

Using the metric tensor $g^{\mu\nu}$ in special relativity:

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad (18)$$

we can calculate the d'Alembert operator as follows:

$$\partial^\mu \partial_\mu = g^{\mu\nu} \partial_\mu \partial_\nu \quad (19)$$

$$= (1 \cdot \partial_t \partial_t - (-1) \cdot \nabla_x^2 - (-1) \cdot \nabla_y^2 - (-1) \cdot \nabla_z^2) \quad (20)$$

$$= \partial_t^2 - (\nabla_x^2 + \nabla_y^2 + \nabla_z^2) \quad (21)$$

$$= \partial_t^2 - \nabla^2, \quad (22)$$

where ∇^2 is the Laplacian operator.

Now let's check if the d'Alembert operator is a Lorentz scalar. Recall that a Lorentz scalar is a quantity that remains invariant under Lorentz transformations. To verify this, we need to consider how the partial derivatives transform under a Lorentz transformation.

Under a Lorentz transformation, the coordinates transform as

$$\tilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu. \quad (23)$$

Applying the chain rule to the partial derivatives, we find

$$\frac{\partial}{\partial \tilde{x}^\mu} = \frac{\partial x^\nu}{\partial \tilde{x}^\mu} \frac{\partial}{\partial x^\nu} = \Lambda_\mu{}^\nu \partial_\nu. \quad (24)$$

Therefore, the d'Alembert operator in the transformed frame can be written as

$$\tilde{\partial}^\mu \tilde{\partial}_\mu = \Lambda^\mu{}_\rho \Lambda_\mu{}^\nu \partial_\rho \partial_\nu. \quad (25)$$

Using the property of Lorentz transformations,

$$\Lambda^\mu{}_\rho \Lambda_\mu{}^\nu = \delta_\rho^\nu, \quad (26)$$

we find that

$$\tilde{\partial}^\mu \tilde{\partial}_\mu = \delta_\rho^\nu \partial_\rho \partial_\nu = \partial^\nu \partial_\nu = \partial^\mu \partial_\mu. \quad (27)$$

This result shows that the d'Alembert operator is indeed a Lorentz scalar, as it remains invariant under Lorentz transformations.

- c) In a covariant formulation of electromagnetism, the electric and magnetic field \mathbf{E} and \mathbf{B} , respectively, are given by the antisymmetric field tensor

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \quad (28)$$

$$F^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \quad (29)$$

The fields can also be described by the four-potential $A_\mu = (\Phi, -\mathbf{A})$, where Φ is a scalar potential and \mathbf{A} is a vector potential.

Show that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ reproduces the fields \mathbf{E} and \mathbf{B} .

We are given that the antisymmetric field tensor $F_{\mu\nu}$ is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (30)$$

To show that this definition reproduces the electric and magnetic fields \mathbf{E} and \mathbf{B} , we will calculate the individual components of $F_{\mu\nu}$ using the four-potential $A_\mu = (\Phi, -\mathbf{A})$.

Let's begin with the time components:

1. $F_{01} = \partial_0 A_1 - \partial_1 A_0 = -\frac{\partial\Phi}{\partial t} - \frac{\partial A_x}{\partial x} = E_x.$
2. $F_{02} = \partial_0 A_2 - \partial_2 A_0 = -\frac{\partial\Phi}{\partial t} - \frac{\partial A_y}{\partial y} = E_y.$
3. $F_{03} = \partial_0 A_3 - \partial_3 A_0 = -\frac{\partial\Phi}{\partial t} - \frac{\partial A_z}{\partial z} = E_z.$

Now let's consider the spatial components:

1. $F_{23} = \partial_2 A_3 - \partial_3 A_2 = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = B_x.$
2. $F_{31} = \partial_3 A_1 - \partial_1 A_3 = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = B_y.$
3. $F_{12} = \partial_1 A_2 - \partial_2 A_1 = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = B_z.$

As you can see, the components of $F_{\mu\nu}$ correspond to the electric and magnetic fields as expected:

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}. \quad (31)$$

This demonstrates that the definition $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ indeed reproduces the fields \mathbf{E} and \mathbf{B} .

- d) Show that $F_{\mu\nu}$ is invariant under the gauge transformation $\tilde{A}_\mu = A_\mu - \partial_\mu f$, where f is an arbitrary function.

We want to show that $F_{\mu\nu}$ is invariant under the gauge transformation $\tilde{A}_\mu = A_\mu - \partial_\mu f$, where f is an arbitrary function. In other words, we need to prove that $\tilde{F}_{\mu\nu} = F_{\mu\nu}$, where $\tilde{F}_{\mu\nu}$ is the field tensor constructed from the gauge-transformed four-potential \tilde{A}_μ .

First, let's compute the components of $\tilde{F}_{\mu\nu}$:

$$\tilde{F}_{\mu\nu} = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu. \quad (32)$$

Substitute the gauge transformation $\tilde{A}_\mu = A_\mu - \partial_\mu f$:

$$\tilde{F}_{\mu\nu} = \partial_\mu (A_\nu - \partial_\nu f) - \partial_\nu (A_\mu - \partial_\mu f) \quad (33)$$

$$= \partial_\mu A_\nu - \partial_\mu \partial_\nu f - \partial_\nu A_\mu + \partial_\nu \partial_\mu f \quad (34)$$

$$= (\partial_\mu A_\nu - \partial_\nu A_\mu) + (\partial_\nu \partial_\mu f - \partial_\mu \partial_\nu f). \quad (35)$$

Now we have

$$\tilde{F}_{\mu\nu} = F_{\mu\nu} + (\partial_\nu \partial_\mu f - \partial_\mu \partial_\nu f). \quad (36)$$

To show that $\tilde{F}_{\mu\nu} = F_{\mu\nu}$, we need to prove that the term $(\partial_\nu \partial_\mu f - \partial_\mu \partial_\nu f)$ vanishes. Note that partial derivatives commute, which means that $\partial_\mu \partial_\nu = \partial_\nu \partial_\mu$. Therefore,

$$\partial_\nu \partial_\mu f - \partial_\mu \partial_\nu f = 0. \quad (37)$$

Consequently, we have

$$\tilde{F}_{\mu\nu} = F_{\mu\nu}. \quad (38)$$

This result shows that the field tensor $F_{\mu\nu}$ is invariant under the gauge transformation $\tilde{A}_\mu = A_\mu - \partial_\mu f$.

- e) Show that in Lorenz gauge, $\partial_\nu A^\nu = 0$, and for no external sources, the Maxwell equations $\partial_\mu F^{\mu\nu} = 0$ reduce to $\partial^\mu \partial_\mu A^\nu = 0$.

In the Lorenz gauge, we have the following condition:

$$\partial_\nu A^\nu = 0. \quad (39)$$

The Maxwell equations in covariant form, in the absence of external sources, are given by:

$$\partial_\mu F^{\mu\nu} = 0. \quad (40)$$

Recall that the field tensor $F^{\mu\nu}$ is defined as:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \quad (41)$$

Let's substitute this expression into the Maxwell equations:

$$\partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = 0. \quad (42)$$

Now, let's apply the product rule for differentiation:

$$(\partial_\mu \partial^\mu) A^\nu - (\partial_\mu \partial^\nu) A^\mu = 0. \quad (43)$$

We want to show that this equation reduces to $\partial^\mu \partial_\mu A^\nu = 0$. To do this, let's examine the second term in the equation above. Using the Lorenz gauge condition $\partial_\nu A^\nu = 0$, we have:

$$\partial_\mu \partial^\nu A^\mu = \partial^\nu (\partial_\mu A^\mu) \quad (44)$$

$$= \partial^\nu (0) \quad (45)$$

$$= 0. \quad (46)$$

Thus, the Maxwell equations in the Lorenz gauge and for no external sources reduce to:

$$\partial^\mu \partial_\mu A^\nu = 0. \tag{47}$$

This is the desired result, showing that the Maxwell equations reduce to the wave equation for the four-potential A^ν in the Lorenz gauge and in the absence of external sources.