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Problem 13.1: Gupta-Bleuler quantization of the radiation field [**Oral** | 6 (+2 bonus) pt(s)]

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> **Learning objective**
>
> In this problem you quantize the radiation field using the *Gupta-Bleuler formalism*. This method is an alternative to the path integral quantization discussed in the lecture, and the canonical quantization in Coulomb gauge that you (should) have seen in your advanced quantum mechanics course. Like the path integral quantization (and unlike the Coulomb gauge quantization), the Gupta-Bleuler procedure is manifestly Lorentz covariant and therefore suited for relativistic quantum field theories. As a final result, you derive the propagator of the radiation field to verify the result obtained in the lecture.

Recall from Problem 1.3 that the Lagrangian density of the radiation field is given by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \,, \tag{1}$$

and the resulting equations of motion are the Maxwell equations.

Using A_μ as our dynamical variable, convince yourself, that the conjugate momentum $\pi^0 = \frac{\partial \mathcal{L}}{\partial(\partial_t A_0)}$ vanishes. Therefore, we cannot quantize the Lagrangian (1) in a Lorentz invariant way (why?).

To overcome this problem, we will use the *Gupta-Bleuler formalism* and choose a new Lagrangian density

$$\mathcal{L}_\xi = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} \left(\partial_\mu A^\mu\right)^2 \,. \tag{2}$$

Note: In Lorenz gauge $\partial_\mu A^\mu = 0$ we recover the Lagrangian density (1) and therefore also the Maxwell equations.

a) Is the new Lagrangian $\mathcal{L}_\xi$ gauge invariant? Derive the equations of motion for the Lagrangian density (2) and the conjugate momentum π^μ. **1pt(s)**

For simplicity, we will restrict ourselves to the case $\xi = 1$ (this is called the Feynman "gauge").

We want to quantize the theory by imposing the canonical commutation relations

$$\begin{aligned} &[A_\mu(t,\boldsymbol{x}), A_\nu(t,\boldsymbol{y})] = 0 = [\pi^\mu(t,\boldsymbol{x}), \pi^\nu(t,\boldsymbol{y})] \,, \\ &[A_\mu(t,\boldsymbol{y}), \pi^\nu(t,\boldsymbol{x})] = i\delta^\nu_\mu \delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \,. \end{aligned} \tag{3}$$

(Why do we use the field π^ν and not π_ν in the commutators?)

*b) Show that the commutation relations (3) can be rewritten as **+1pt(s)**

$$\left[A_\mu(t,\boldsymbol{y}), \dot{A}^\nu(t,\boldsymbol{x})\right] = -i\delta^\nu_\mu \delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \quad \text{and} \quad \left[\dot{A}^\mu(t,\boldsymbol{x}), \dot{A}^\nu(t,\boldsymbol{y})\right] = 0 \,. \tag{4}$$

c) Show that the mode expansion **1pt(s)**

$$A_\mu(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\,|\boldsymbol{p}|}} \sum_{r=0}^{3} \left[a^r_{\boldsymbol{p}} \epsilon^r_\mu(p) e^{-ipx} + a^{r\dagger}_{\boldsymbol{p}} \epsilon^{r\,*}_\mu(p) e^{+ipx} \right] , \tag{5}$$

fulfills the commutation relations (4), provided that the modes satisfy the bosonic commutation relations

$$\left[a^r_{\boldsymbol{p}}, a^s_{\boldsymbol{q}}\right] = 0 \quad \text{and} \quad \left[a^r_{\boldsymbol{p}}, a^{s\dagger}_{\boldsymbol{q}}\right] = -g^{rs} \, (2\pi)^3 \, \delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}) \ . \tag{6}$$

Here we include *four* possible photon polarization vectors $\epsilon^r(p)$ with $r \in \{0, 1, 2, 3\}$ and introduced the four-vector $p^\mu = (|\boldsymbol{p}|\, , \boldsymbol{p})$.

Hint:

Without loss of generality, we can fix a polarization basis for each momentum $\boldsymbol{p} = |\boldsymbol{p}|\, \hat{\boldsymbol{p}}$. We choose the polarization vectors $\epsilon^r(p)$ as follows:

- Scalar polarization: $\epsilon^0_\mu(p) \equiv (1, 0, 0, 0)$,
- Longitudinal polarization: $\epsilon^3_\mu(p) \equiv (0, -\hat{\boldsymbol{p}})$,
- Transversal polarizations: $\epsilon^{1,2}_\mu(p) \equiv (0, -\hat{\boldsymbol{p}}_{\perp 1,2})$, with $\hat{\boldsymbol{p}}_{\perp 1,2}$ being orthogonal to $\hat{\boldsymbol{p}}$ and to each other.

With these definitions, show that $\epsilon^r(p) \cdot \epsilon^{s\,*}(p) = g^{rs}$ and prove the relation

$$\sum_r \epsilon^r_\mu(p) \epsilon^{r\,*}_\nu(p) g^{rr} = g_{\mu\nu} \, . \tag{7}$$

d) Calculate the norm of the single photon states **1pt(s)**

$$|1_r\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\,|\boldsymbol{p}|}} f(\boldsymbol{p}) a^{r\dagger}_{\boldsymbol{p}} \left|0\right\rangle \ , \tag{8}$$

for an arbitrary (normalizable) function $f(\boldsymbol{p})$ and show that the Fock space of the photons has an indefinite metric. Finally, evaluate the expression

$$\left(\int \frac{d^3p}{(2\pi)^3} a^{0\dagger}_{\boldsymbol{p}} a^0_{\boldsymbol{p}} \right) |1_0\rangle \tag{9}$$

and use the result to define a number operator $\hat{N}_r$ that counts the number of photons with polarization $r \in \{0, 1, 2, 3\}$.

Having an indefinite metric – and therefore negative probabilities – spoils the formalism of quantum mechanics. Furthermore, so far we have *four* degrees of freedom for the photon, but expect only *two* transversal polarizations for physical photons. As it turns out, those problems are linked and can be fixed together, as we will see in the following.

Classically, to recover the original Lagrangian density Eq. (1) and the Maxwell equations from the modified action Eq. (2), we would like to impose the Lorenz gauge condition $\partial_\mu A^\mu = 0$.

This, however, is not possible on the level of *operators* A^μ (Why?).

Instead, we could try to impose the Lorenz gauge condition on the level of *expectation values* for *physical states* $|\psi\rangle$, i.e., $\langle\psi|\partial A|\psi\rangle \stackrel{!}{=} 0$; let $\mathcal{H}_1$ denote the space of such physical states. $\mathcal{H}_1$ should be a linear (Hilbert) space, which can be satisfied by the stronger condition

$$\partial^\mu A^{(+)}_\mu \left|\psi\right\rangle \stackrel{!}{=} 0 \quad \text{for all} \quad \left|\psi\right\rangle \in \mathcal{H}_1 \, . \tag{10}$$

Here $A^{(+)}_\mu$ is the annihilation (positive frequency) part of A_μ.

e) First, prove that condition (10) spans a linear subspace $\mathcal{H}_1$ of the Fock space $\mathcal{H}$. **1**pt(s)

Next, show that for our choice of polarization vectors $\epsilon^r_\mu(p)$, Eq. (10) is equivalent to

$$\sum_{r=0,3} p^\mu \epsilon^r_\mu(p) a^r_{\boldsymbol{p}} |\psi\rangle = 0 \quad \text{and therefore} \quad \left(a^0_{\boldsymbol{p}} - a^3_{\boldsymbol{p}}\right)|\psi\rangle = 0\,. \tag{11}$$

Explain why we can henceforth focus on states $|\phi\rangle$ with only scalar and longitudinal excitations to examine this condition further.

We now consider states $|\phi_n\rangle \in \mathcal{H}_1$ with in total n (scalar or longitudinal) photons, i.e.,

$$\hat{N}_{0+3}|\phi_n\rangle = \int \frac{d^3p}{(2\pi)^3}\left[a^{3\dagger}_{\boldsymbol{p}} a^3_{\boldsymbol{p}} - a^{0\dagger}_{\boldsymbol{p}} a^0_{\boldsymbol{p}}\right]|\phi_n\rangle = n\,|\phi_n\rangle\ , \tag{12}$$

where $\hat{N}_{0+3}$ is the total number operator for scalar and longitudinal photons (Explain the signs!). A general state $|\phi\rangle$ can then be written as linear combination of states $|\phi_n\rangle$ with different photon numbers,

$$|\phi\rangle = \sum_n c_n |\phi_n\rangle\ , \tag{13}$$

where c_n are some coefficients.

f) Show that $\left\langle \phi_n \middle| \hat{N}_{0+3} \middle| \phi_n \right\rangle = 0$. What does this imply for the norm of $|\phi_n\rangle$ and $|\phi\rangle$? **1**pt(s)

*g) So far the coefficients c_n are arbitrary. Show that the energy $\frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}$ does not depend on these coefficients, where the Hamiltonian is given by **+1**pt(s)

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_{r=0}^{3} (-g^{rr}) p^0 a^{r\dagger}_{\boldsymbol{p}} a^r_{\boldsymbol{p}}\,. \tag{14}$$

Indeed, *all* physical observables are independent of the coefficients c_n. The arbitrariness of $|\phi\rangle$ reflects the gauge freedom of A^μ and does not have physical effects.

h) Finally, calculate the photon propagator **1**pt(s)

$$\langle 0|\mathcal{T} A_\mu(x) A_\nu(y)|0\rangle\ . \tag{15}$$

Hint: Similar to the Feynman propagator of the Klein-Gordon field derived in the lecture, use the residue theorem to transform the 3D momentum integral into a 4D momentum integral $\int \frac{d^3p}{(2\pi)^3} \to \int \frac{d^4p}{(2\pi)^4}$.